

Hydrogen-like atoms in relativistic QED

MARTIN KÖNENBERG

Fakultät für Mathematik und Informatik, FernUniversität Hagen,
Lützowstraße 125, D-58084 Hagen, Germany.

Present address:

Fakultät für Physik, Universität Wien,
Boltzmanngasse 5, 1090 Vienna, Austria.
`martin.koenenberga@univie.ac.at`

OLIVER MATTE

Institut für Mathematik, TU Clausthal,
Erzstraße 1, D-38678 Clausthal-Zellerfeld, Germany.

Present address:

Institut for Matematik, Århus Universitet,
Ny Munkegade 118, DK-8000 Århus, Denmark.
`matte@math.lmu.de`

EDGARDO STOCKMEYER

Mathematisches Institut, Ludwig-Maximilians-Universität,
Theresienstraße 39, D-80333 München, Germany.
`stock@math.lmu.de`

Abstract

In this review we consider two different models of a hydrogenic atom in a quantized electromagnetic field that treat the electron relativistically. The first one is a no-pair model in the free picture, the second one is given by the semi-relativistic Pauli-Fierz Hamiltonian. For both models we discuss the semi-boundedness of the Hamiltonian, the strict positivity of the ionization energy, and the exponential localization in position space of spectral subspaces corresponding to energies below the ionization threshold. Moreover, we prove the existence of degenerate ground state eigenvalues at the bottom of the spectrum of the Hamiltonian in both models. All these results hold true, for arbitrary values of the fine-structure constant, e^2 , and the ultra-violet cut-off, and for a general class of electrostatic potentials including the Coulomb potential with nuclear charges less than (sometimes including) the critical charges without radiation field, namely $e^{-2}2/\pi$ for the semi-relativistic Pauli-Fierz operator and $e^{-2}2/(2/\pi + \pi/2)$ for the no-pair operator. Apart from a detailed discussion of diamagnetic inequalities in QED (which are applied to study the semi-boundedness) all results stem from earlier articles written by the authors. While a few proofs are merely sketched, we streamline earlier proofs or present alternative arguments at many places.

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1 Introduction

In the late 90's and the past decade the existence of ground states of atoms and molecules interacting with the quantized photon field has been intensively studied by mathematicians in the framework of non-relativistic quantum electrodynamics (QED). The corresponding Hamiltonian is the *non-relativistic Pauli-Fierz operator* which, in the case of a hydrogen-like atom, is given as

$$H_{\gamma}^{\text{nr}} := (\boldsymbol{\sigma} \cdot (-i\nabla_{\mathbf{x}} + \mathbf{A}))^2 - \frac{\gamma}{|\mathbf{x}|} + H_{\text{f}}. \quad (1.1)$$

Here $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is a vector containing the Pauli spin matrices, \mathbf{A} is the quantized vector potential in the Coulomb gauge, and H_f is the energy of the photon field. The symbol \mathbf{A} includes a prefactor entering into the analysis as a parameter, namely the square-root of the fine structure constant which equals the elementary charge, $e > 0$, in the units chosen in this paper. The Coulomb coupling constant, $\gamma > 0$, is the product of e and the nuclear charge. We shall, however, always consider it as an independent parameter since the interrelationship between e and γ does not play any role in our work. \mathbf{A} additionally depends on some ultra-violet cut-off parameter, $\Lambda > 0$. By now it is well-known that H_γ^{nr} has a self-adjoint realization in the Hilbert space $L^2(\mathbb{R}_x^3, \mathbb{C}^2) \otimes \mathcal{F}_b[\mathcal{K}]$ whose spectrum is bounded below. Here $\mathcal{F}_b[\mathcal{K}]$ denotes the bosonic Fock space modeled over the Hilbert space for a single photon, $\mathcal{K} = L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$. Proving the existence of ground states for H_γ^{nr} means to show that the infimum of the spectrum of H_γ^{nr} is an eigenvalue corresponding to some normalizable ground state eigenvector in $L^2(\mathbb{R}_x^3, \mathbb{C}^2) \otimes \mathcal{F}_b[\mathcal{K}]$. Because of the spin degrees of freedom this ground state eigenvalue will be degenerate. Mathematically, the study of the eigenvalue at the bottom of the spectrum of H_γ^{nr} is very subtle because the spectrum of H_γ^{nr} is continuous up to its minimum and the eigenvalue is, thus, not an isolated one. In particular, many standard methods of spectral theory do not apply and several new mathematical techniques had to be invented in order to overcome this problem.

The first proofs of the existence of ground states for H_γ^{nr} and its molecular analogs have been given in [4, 6], for small values of the involved parameters e and Λ . A few years later the existence of ground states for a molecular non-relativistic Pauli-Fierz Hamiltonian has been established, for arbitrary values of e and Λ , in [17] by means of a certain binding condition which has been verified later on in [33]. Moreover, infra-red finite algorithms and spectral theoretic renormalization group methods have been applied to various models of non-relativistic QED to study their ground state energies and projections [2, 3, 4, 5, 6, 7, 15]. These sophisticated methods yield very precise results as they rely on constructive algorithms rather than on compactness arguments as in [4, 6, 17]. They work, however, only in a regime where e and Λ are sufficiently small.

A question which arises naturally in this context is whether these results still hold true when the electrons are described by a relativistic operator. In this review, which summarizes results from [25, 28, 29, 36, 37, 48], we give a positive answer to this question. We study two different models that seem to be natural candidates for a mathematical analysis: The first one is given by the following no-pair operator,

$$H_\gamma^{\text{np}} := P_{\mathbf{A}}^+ \left(D_{\mathbf{A}} - \frac{\gamma}{|\mathbf{x}|} + H_f \right) P_{\mathbf{A}}^+, \quad (1.2)$$

or more generally,

$$H_V^{\text{np}} := P_{\mathbf{A}}^+ \left(D_{\mathbf{A}} + V + H_f \right) P_{\mathbf{A}}^+, \quad (1.3)$$

for some electrostatic potential V . Here $D_{\mathbf{A}}$ is the free Dirac operator minimally coupled to \mathbf{A} and $P_{\mathbf{A}}^+$ denotes the spectral projection onto the positive spectral

subspace of $D_{\mathbf{A}}$,

$$P_{\mathbf{A}}^+ := \mathbb{1}_{(0,\infty)}(D_{\mathbf{A}}).$$

The no-pair operator is considered as an operator acting in the projected Hilbert space, $\mathcal{H}_{\mathbf{A}}^+ := P_{\mathbf{A}}^+ \mathcal{H}$. It is thus acting on a space where the electron and photon degrees of freedom are always linked together. The analog of H_{γ}^{np} for molecules has been introduced in [32] as a mathematical model to study the stability of matter in relativistic QED. Under certain restrictions on e , Λ , and the nuclear charges it has been shown in [32] that the quadratic form of a molecular no-pair operator is bounded from below by some constant which is proportional to the number of involved electrons and nuclei and uniform in the nuclear positions. Moreover, the (positive) binding energy has been estimated from above in [31]. In fact, there are numerous mathematical contributions on no-pair models where magnetic fields are not taken into account or treated classically; see, e.g., [39] for a list of references. For instance, it is shown in [13] that a no-pair operator with Coulomb potential but without quantized photon field – which is then often called the Brown-Ravenhall operator – has a critical coupling constant, $\gamma_c^{\text{np}} := 2/(2/\pi + \pi/2)$, such that the corresponding quadratic form becomes unbounded below when γ exceeds this value. Moreover, various molecular no-pair models (without quantized fields, however) are widely used in quantum chemistry and in the theoretical and numerical study of highly ionized heavy atoms; see, e.g., [10, 22, 46]. In this context several different choices of the projections determining the model find their applications. For instance, one can include the Coulomb or a Hartree-Fock potential in the projection. These two choices are covered by the results of [39] where two of the present authors provide a spectral analysis of a class of molecular no-pair Hamiltonians with classical magnetic fields. The choice of the projection $P_{\mathbf{A}}^+$ which does not contain any potential terms is referred to as the free picture. We remark that it is essential to include the vector potential in the projection determining the no-pair model. For, if $P_{\mathbf{A}}^+$ is replaced by $P_{\mathbf{0}}^+$, then the analog of (1.2) describing N interacting electrons becomes unstable as soon as $N \geq 2$ [18, 32, 34]. Moreover, the operator in (1.2) is formally gauge invariant and this would not hold true anymore with $P_{\mathbf{0}}^+$ in place of $P_{\mathbf{A}}^+$. Gauge invariance plays, however, an important role in the proof of the existence of ground states as it permits to derive bounds on the number of soft photons. In fact, employing a mild infra-red regularization it is possible to prove the existence of ground states for the operator in (1.2) with $P_{\mathbf{A}}^+$ replaced by $P_{\mathbf{0}}^+$ [24, 35]. It seems, however, unlikely that the infra-red regularization can be dropped in this case [24].

The second operator treated in this review, the semi-relativistic Pauli-Fierz operator, is given as

$$\sqrt{(\boldsymbol{\sigma} \cdot (-i\nabla + \mathbf{A}))^2 + \mathbb{1}} - \frac{\gamma}{|\mathbf{x}|} + H_{\text{f}}, \quad (1.4)$$

where $\boldsymbol{\sigma}$ is a vector containing the Pauli spin matrices. Since $\sqrt{-\Delta}$ and $1/|\mathbf{x}|$ both scale as one over the length there will again be some critical upper bound on all values of $\gamma > 0$ for which (1.4) defines a semi-bounded quadratic form. As we shall see this upper bound is at least as big as (in fact equal to [28])

the critical constant in Kato's inequality, $\gamma_c^{\text{PF}} := 2/\pi$. Again we shall study the semi-relativistic Pauli-Fierz operator also for a more general class of electrostatic potentials V . The latter (straightforward) generalization is relevant in a forthcoming work of the first two authors devoted to the enhanced binding effect [27].

Also the semi-relativistic Pauli-Fierz operator has been investigated earlier in a few mathematical articles. For instance it appears in the mathematical study of Rayleigh scattering [14] where the finite propagation speed of relativistic particles turns out to be an advantageous feature in comparison to models of non-relativistic QED. (The electron spin has, however, been neglected in [14].) For $\gamma = 0$, the fiber decomposition of (1.4) with respect to different values of the total momentum has been studied in [40]. Moreover, there is a remark in [40] relevant for us saying that every (speculative) eigenvalue of the operator in (1.4) is at least doubly degenerate since the Hamiltonian commutes with some anti-linear involution. The existence of the renormalized electron mass in the semi-relativistic Pauli-Fierz model, i.e. twice continuous differentiability of the mass shell in balls about zero, is proved in [26], for small values of e . The last author has shown [48] that, when the speed of light, c , is re-introduced as a parameter and $\gamma \in [0, \gamma_c^{\text{PF}})$, then the operator in (1.4) converges in norm resolvent sense to the non-relativistic Pauli-Fierz operator in (1.1), as c tends to infinity. Finally, there is a contribution [21] on the existence of binding in the semi-relativistic Pauli-Fierz model; see Remark 4.2.

We should also mention that the existence of ground states in a relativistic model describing both the photons and the electrons and positrons by quantized fields has been studied mathematically in [8]. To this end infra-red and ultra-violet cut-offs for the momenta of all involved particles are imposed in the interaction term of the Hamiltonian considered in [8].

In the remaining part of this introduction we explain the organization of this review article and summarize briefly our main results. In Section 2 we recall the definitions of some operators appearing in QED and introduce the no-pair and semi-relativistic Pauli-Fierz Hamiltonians more precisely. Although the general strategy of our whole project relies on the methods developed in [4, 6, 17] the spectral analysis of the operators treated in this article poses a variety of new and non-trivial mathematical obstacles which is mainly caused by their non-locality. In fact, both operators do not act as partial differential operators on the electronic degrees of freedom anymore as it is the case in non-relativistic QED. In this respect the no-pair operator is harder to analyze than the semi-relativistic Pauli-Fierz operator since also the electrostatic potential and the radiation field energy become non-local due to the presence of the spectral projections $P_{\mathbf{A}}^+$. In the last subsection of Section 2 we explain a few mathematical tools used to overcome some of the problems posed by the non-locality thus preparing the reader for the proofs in the succeeding sections.

In Section 3 we provide some basic relative bounds and study the semi-boundedness of the Hamiltonian in both models under consideration. We start with a discussion of various diamagnetic inequalities for quantized vector potentials. They are employed to prove that the quadratic form of the semi-relativistic

Pauli-Fierz operator is semi-bounded below on some natural dense domain, for a suitable class of potentials including the Coulomb potential with coupling constants $\gamma \in [0, \gamma_c^{\text{PF}}]$. For the no-pair operator we obtain similar results with Coulomb coupling constants $\gamma \in [0, \gamma_c^{\text{np}}]$. As a consequence, both operators can be realized as self-adjoint operators in a physically distinguished way by means of a Friedrichs extension. We point out that the results on the semi-boundedness, as well as all further results described below hold true, for arbitrary values of e and Λ .

Section 4 is devoted to the study of binding. For both models treated here we show that the infimum of the spectrum of the Hamiltonian with appropriate non-vanishing potential is strictly less than its ionization threshold which, by definition, is equal to the infimum of the spectrum of the Hamiltonian without electrostatic potential. To this end we employ trial functions which are tensor products of electronic and photonic wave functions and work with unitarily equivalent Hamiltonians in order to separate the electronic and photonic degrees of freedom. The unitary transformation used here represents the free Hamiltonian ($V = 0$) as a direct integral of fiber operators with respect to different values of the total momentum.

Typically, proofs of the existence of ground states in QED require some information on the localization of spectral subspaces corresponding to energies below the ionization threshold (or at least of certain approximate ground state eigenfunctions). Here localization is understood with respect to the electron coordinates. We establish this prerequisite in Section 5 by adapting some ideas from [4, 16]. In this section we present streamlined versions of some of our earlier arguments from [37]. Moreover, we implement later improvements [25] on parts of the results of [37] by providing optimized exponential decay rates in the case of the semi-relativistic Pauli-Fierz operator that reduce to the typical relativistic decay rates known for the electronic Dirac and square-root operators, when the radiation field is turned off. The class of potentials allowed for in Section 5 covers Coulomb potentials with $\gamma \in [0, \gamma_c^{\text{PF}}]$ in the case of the semi-relativistic Pauli-Fierz model and with $\gamma \in [0, \gamma_c^{\text{np}}]$ in the no-pair model. It is, however, possible to prove the exponential localization for the no-pair operator with Coulomb potential also in the critical case $\gamma = \gamma_c^{\text{np}}$ by another modification of the arguments [25].

The main results of our joint project are the proofs of the existence of ground states for the semi-relativistic Pauli-Fierz and no-pair operators. As already stressed above our proofs work, for arbitrary values of e and Λ and for a class of potentials including the Coulomb potential with $\gamma \in (0, \gamma_c^{\text{PF}})$ and $\gamma \in (0, \gamma_c^{\text{np}})$, respectively. Starting from these results it is actually possible to prove the existence of ground states also in the critical cases $\gamma = \gamma_c^{\text{PF}}$ and $\gamma = \gamma_c^{\text{np}}$, respectively, by means of an additional approximation argument. We refrain from explaining any details of the latter in the present article and refer the interested reader to [25] instead.

The proofs of the existence of ground states given here are divided into two steps:

First, one introduces a photon mass, $m > 0$, and shows that the resulting

Hamiltonians possess normalized ground state eigenfunctions, ϕ_m [4, 6, 17]. In this first step, which is presented in Section 6, we employ a discretization of the photon momenta as in [6]. Roughly speaking, by discretizing the photon momenta one may replace the Fock space $\mathcal{F}_b[\mathcal{H}]$ by a Fock space modeled over some ℓ^2 space. As a consequence the spectrum of the radiation field energy becomes discrete and one can in fact argue that the total Hamiltonian has discrete eigenvalues at the bottom of its spectrum when all small photon momenta are discarded. At this point we add a new observation based on the localization estimates to the arguments of [6] which allows to carry through the proof, for all values of e and Λ . (In [4, 6] these parameters were assumed to be sufficiently small.) Another technical tool turns out to be very helpful in order to compare discretized and non-discretized Hamiltonians (or those with and without photon mass), namely, certain higher order estimates allowing to control higher powers of the radiation field energy by corresponding powers of the resolvent of the total Hamiltonian. For the semi-relativistic Pauli-Fierz operator such estimates have been established in [14]. In [36] one of the present authors re-proves the higher order estimates from [14] by means of a different and more model-independent method which also permits to derive higher order estimates for a (molecular) no-pair operator for the first time. We discuss these higher order estimates in Subsection 6.4 but refrain from repeating their proofs. We remark that many of the arguments presented in Section 6 are alternatives to those used in [28, 29].

The second step in the proof of the existence of ground states comprises of a compactness argument showing that every sequence $\{\phi_{m_j}\}$ with $m_j \searrow 0$ contains a strongly convergent subsequence. In fact, one readily verifies that the limit of such a subsequence is a ground state eigenfunction of the original Hamiltonian with massless photons. This step is performed in Section 8, in parts by means of arguments alternative to those in [28, 29]. The compactness argument requires, however, a number of non-trivial ingredients. First, we need two infra-red bounds, namely a bound on the number of photons with low energy in the eigenfunctions ϕ_m (soft photon bound) [6, 17] and a certain bound on the weak derivatives of ϕ_m with respect to the photon momenta (photon derivative bound) [17]. To derive the infra-red bounds one can either adapt a procedure proposed in [17] (this is carried through in earlier preprint versions of [28, 29] available on the arXiv) or establish a formula for $a(k)\phi_m$ by means of a virial type argument and infer the bounds from that representation. We outline the proof of the latter formula and of the soft photon bound for the semi-relativistic Pauli-Fierz operator in Section 7. The photon derivative bound and the infra-red bounds for the no-pair operator are derived by very similar procedures and we refer the interested reader to our original articles [28, 29] for the rather dull technical details. The arguments presented in Section 7 are also intended to emphasize the role of the gauge invariance of the models treated here. In fact, one first applies a unitary operator-valued gauge transformation (Pauli-Fierz transformation) and the infra-red bounds are then derived in the new gauge. Without the gauge transformation one would encounter infra-red divergent integrals.

As soon as the infra-red estimates are established, the soft photon bound

and the exponential localization estimates show that the eigenvectors ϕ_m are localized uniformly in m with respect to the electron and photon coordinates and that their components in all but finitely many Fock space sectors are negligible. Moreover, the photon derivative bound implies that their weak derivatives with respect to the photon momenta are uniformly bounded in a suitable L^p -space and since their energies are uniformly bounded we also know that the vectors have uniformly bounded half-derivatives with respect to the electron coordinates in L^2 . It is an idea of [17] to exploit such information by applying compact embedding theorems for Sobolev-type spaces to single out subsequences that converge strongly in L^2 . In the semi-relativistic setting considered in Section 8 some classical embedding theorems by Nikol'skiĭ turn out to be useful substitutes for the Rellich-Kondrachov theorem employed in [17]. At the end of Section 8 we also show that the ground state energies of both Hamiltonians are degenerate eigenvalues.

At last, in Section 9, we present the proofs of some technical results we have referred to in earlier sections so that most parts of this review become essentially self-contained.

2 Definition of the models

In order to introduce the models treated in this article more precisely we first fix our notation and recall some standard facts.

2.1 Operators in Fock-space

The state space of the quantized photon field is the bosonic Fock space,

$$\mathcal{F}_b[\mathcal{K}] := \bigoplus_{n=0}^{\infty} \mathcal{F}_b^{(n)}[\mathcal{K}] \ni \psi = (\psi^{(0)}, \psi^{(1)}, \psi^{(2)}, \dots).$$

It is modeled over the one photon Hilbert space

$$\mathcal{K} := L^2(\mathbb{R}^3 \times \mathbb{Z}_2, dk), \quad \int dk := \sum_{\lambda \in \mathbb{Z}_2} \int_{\mathbb{R}^3} d^3 \mathbf{k}.$$

$k = (\mathbf{k}, \lambda)$ denotes a tuple consisting of a photon wave vector, $\mathbf{k} \in \mathbb{R}^3$, and a polarization label, $\lambda \in \mathbb{Z}_2$. Moreover, $\mathcal{F}_b^{(0)}[\mathcal{K}] := \mathbb{C}$ and $\mathcal{F}_b^{(n)}[\mathcal{K}]$ is the subspace of all complex-valued, square integrable functions on $(\mathbb{R}^3 \times \mathbb{Z}_2)^n$ that remain invariant under permutations of the $n \in \mathbb{N}$ wave vector/polarization tuples. The subspace

$$\mathcal{C}_0 := \mathbb{C} \oplus \bigoplus_{n \in \mathbb{N}} C_0((\mathbb{R}^3 \times \mathbb{Z}_2)^n) \cap \mathcal{F}_b^{(n)}[\mathcal{K}] \quad (\text{Algebraic direct sum.})$$

is dense in $\mathcal{F}_b[\mathcal{K}]$. The energy of the photon field, $H_f := d\Gamma(\omega)$, is given as the second quantization of the dispersion relation $\omega(k) := |\mathbf{k}|$, $k = (\mathbf{k}, \lambda) \in \mathbb{R}^3 \times \mathbb{Z}_2$.

We recall that the second quantization of some real-valued Borel measurable function ϖ is given by $(d\Gamma(\varpi)\psi)^{(0)} = 0$ and

$$(d\Gamma(\varpi)\psi)^{(n)}(k_1, \dots, k_n) = \sum_{j=1}^n \varpi(k_j) \psi^{(n)}(k_1, \dots, k_n), \quad n \in \mathbb{N},$$

for all $\psi = (\psi^{(n)})_{n=0}^\infty \in \mathcal{F}_b[\mathcal{K}]$ such that $([d\Gamma(\varpi)\psi]^{(n)})_{n=0}^\infty \in \mathcal{F}_b[\mathcal{K}]$. We further recall that the creation and the annihilation operators of a photon state $f \in \mathcal{K}$ are given, for $n \in \mathbb{N}$, by

$$\begin{aligned} (a^\dagger(f)\psi)^{(n)}(k_1, \dots, k_n) &= n^{-1/2} \sum_{j=1}^n f(k_j) \psi^{(n-1)}(\dots, k_{j-1}, k_{j+1}, \dots), \\ (a(f)\psi)^{(n-1)}(k_1, \dots, k_{n-1}) &= n^{1/2} \int \bar{f}(k) \psi^{(n)}(k, k_1, \dots, k_{n-1}) dk, \end{aligned}$$

and $(a^\dagger(f)\psi)^{(0)} = 0$, $a(f)\Omega = 0$, where $\Omega := (1, 0, 0, \dots) \in \mathcal{F}_b[\mathcal{K}]$ is the vacuum vector. We define $a^\dagger(f)$ and $a(f)$ on their maximal domains. For $f, g \in \mathcal{K}$, the following canonical commutation relations hold true on \mathcal{C}_0 ,

$$[a(f), a(g)] = [a^\dagger(f), a^\dagger(g)] = 0, \quad [a(f), a^\dagger(g)] = \langle f | g \rangle \mathbb{1}.$$

For a three-vector of functions $\mathbf{f} = (f^{(1)}, f^{(2)}, f^{(3)}) \in \mathcal{K}^3$, the symbol $a^\sharp(\mathbf{f})$ denotes the triple of operators $a^\sharp(\mathbf{f}) := (a^\sharp(f^{(1)}), a^\sharp(f^{(2)}), a^\sharp(f^{(3)}))$, where a^\sharp is always either a or a^\dagger .

2.2 Interaction term

Next, we describe the interaction between four-spinors and the photon field. The full Hilbert space underlying our models is

$$\mathcal{H} := L^2(\mathbb{R}_x^3, \mathbb{C}^4) \otimes \mathcal{F}_b[\mathcal{K}].$$

It contains the dense subspace

$$\mathcal{D} := C_0^\infty(\mathbb{R}_x^3, \mathbb{C}^4) \otimes \mathcal{C}_0. \quad (\text{Algebraic tensor product.})$$

We introduce the self-adjoint Dirac matrices $\alpha_1, \alpha_2, \alpha_3$, and β that act on the four spinor components of an element from \mathcal{H} , that is, on the second tensor factor in $\mathcal{H} \cong L^2(\mathbb{R}_x^3) \otimes \mathbb{C}^4 \otimes \mathcal{F}_b[\mathcal{K}]$. They are given by

$$\alpha_j := \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad j \in \{1, 2, 3\}, \quad \beta := \alpha_0 := \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix},$$

where $\sigma_1, \sigma_2, \sigma_3$ denote the standard Pauli matrices, and fulfill the Clifford algebra relations

$$\alpha_i \alpha_j + \alpha_j \alpha_i = 2 \delta_{ij} \mathbb{1}, \quad i, j \in \{0, 1, 2, 3\}. \quad (2.1)$$

The interaction between the electron/positron and photon degrees of freedom in the Coulomb gauge is given as $\boldsymbol{\alpha} \cdot \mathbf{A} := \alpha_1 A^{(1)} + \alpha_2 A^{(2)} + \alpha_3 A^{(3)}$, where

$$\mathbf{A} := (A^{(1)}, A^{(2)}, A^{(3)}) := a^\dagger(\mathbf{G}) + a(\mathbf{G}), \quad a^\dagger(\mathbf{G}) := \int_{\mathbb{R}^3}^\oplus \mathbb{1}_{\mathbb{C}^4} \otimes a^\dagger(\mathbf{G}_{\mathbf{x}}) d^3 \mathbf{x}.$$

The physical choice of the coupling function $\mathbf{G}_{\mathbf{x}} = (G_{\mathbf{x}}^{(1)}, G_{\mathbf{x}}^{(2)}, G_{\mathbf{x}}^{(3)})$ is

$$\mathbf{G}_{\mathbf{x}}(k) := -e \frac{\mathbb{1}_{\{|\mathbf{k}| \leq \Lambda\}}}{2\pi \sqrt{|\mathbf{k}|}} e^{-i\mathbf{k} \cdot \mathbf{x}} \boldsymbol{\varepsilon}(k), \quad (2.2)$$

for $\mathbf{x} \in \mathbb{R}^3$ and almost every $k = (\mathbf{k}, \lambda) \in \mathbb{R}^3 \times \mathbb{Z}_2$. The parameter $\Lambda > 0$ is an ultraviolet cut-off and $e \in \mathbb{R}$. (In nature $e^2 \approx 1/137$ is Sommerfeld's fine-structure constant which equals the square of the elementary charge in our units¹.) The values of e and Λ can be chosen arbitrarily in the whole article. Writing

$$\mathbf{k}_\perp := (k^{(2)}, -k^{(1)}, 0), \quad \mathbf{k} = (k^{(1)}, k^{(2)}, k^{(3)}) \in \mathbb{R}^3, \quad (2.3)$$

the polarization vectors are given as

$$\boldsymbol{\varepsilon}(\mathbf{k}, 0) = \frac{\mathbf{k}_\perp}{|\mathbf{k}_\perp|}, \quad \boldsymbol{\varepsilon}(\mathbf{k}, 1) = \frac{\mathbf{k}}{|\mathbf{k}|} \wedge \boldsymbol{\varepsilon}(\mathbf{k}, 0), \quad (2.4)$$

for $\mathbf{k} \in \mathbb{R}^3 \setminus \{0\}$ with $\mathbf{k}_\perp \neq 0$. It is sufficient to determine $\boldsymbol{\varepsilon}$ almost everywhere. Many of our results and estimates do not depend on the special choice of $\mathbf{G}_{\mathbf{x}}$. If we consider a larger class of coupling functions at a certain point in this review we shall explain the required properties of $\mathbf{G}_{\mathbf{x}}$ at the beginning of the corresponding (sub)section.

2.3 The semi-relativistic Pauli-Fierz and no-pair Hamiltonians

In order to define the no-pair and semi-relativistic Pauli-Fierz operators we recall that the free Dirac operator minimally coupled to \mathbf{A} is given as

$$D_{\mathbf{A}} := \boldsymbol{\alpha} \cdot (-i\nabla + \mathbf{A}) + \beta := \sum_{j=1}^3 \alpha_j (-i\partial_{x_j} + a^\dagger(G_{\mathbf{x}}^{(j)}) + a(G_{\mathbf{x}}^{(j)})) + \beta. \quad (2.5)$$

A straightforward application of Nelson's commutator theorem shows that $D_{\mathbf{A}}$ is essentially self-adjoint on \mathscr{D} ; see, e.g., [32, 40]. We denote its closure starting from \mathscr{D} again by the same symbol. As a consequence of (2.1) we further have

$$D_{\mathbf{A}}^2 = \mathcal{T}_{\mathbf{A}} \oplus \mathcal{T}_{\mathbf{A}}, \quad \mathcal{T}_{\mathbf{A}} := (\boldsymbol{\sigma} \cdot (-i\nabla + \mathbf{A}))^2 + \mathbb{1}, \quad (2.6)$$

¹ Energies are measured in units of mc^2 , m denoting the rest mass of an electron and c the speed of light. Length, i.e. \mathbf{x} , are measured in units of $\hbar/(mc)$, which is the Compton wave length divided by 2π . \hbar is Planck's constant divided by 2π . The photon wave vectors \mathbf{k} are measured in units of 2π times the inverse Compton wavelength, mc/\hbar .

on \mathcal{D} . In particular, $D_{\mathbf{A}}^2 \geq 1$ on \mathcal{D} , and since $D_{\mathbf{A}}$ is essentially self-adjoint on \mathcal{D} we see that $\|(D_{\mathbf{A}} - z)\psi\| \geq (1 - |z|)\|\psi\|$, $\psi \in \mathcal{D}(D_{\mathbf{A}})$, $z \in \mathbb{C}$, whence

$$\sigma(D_{\mathbf{A}}) \subset (-\infty, -1] \cup [1, \infty).$$

Contrary to the usual convention used also in the introduction we define the *semi-relativistic Pauli-Fierz operator* as an operator acting in \mathcal{H} a priori by

$$H_{\gamma}^{\text{PF}} \varphi := (|D_{\mathbf{A}}| - \frac{\gamma}{|\mathbf{x}|} + H_{\text{f}}) \varphi, \quad \varphi \in \mathcal{D}, \quad (2.7)$$

$$H_V^{\text{PF}} \varphi := (|D_{\mathbf{A}}| + V + H_{\text{f}}) \varphi, \quad \varphi \in \mathcal{D}. \quad (2.8)$$

We shall impose appropriate conditions on the general potential $V \in L_{\text{loc}}^2(\mathbb{R}^3, \mathbb{R})$ later on. In fact, the operator defined in (2.7) is a two-fold copy of the one given in (1.4) since $|D_{\mathbf{A}}| = \mathcal{T}_{\mathbf{A}}^{1/2} \oplus \mathcal{T}_{\mathbf{A}}^{1/2}$ by (2.6). We prefer to consider the operator defined by (2.7) to have a unified notation throughout the following sections. The *no-pair operator* acts in a projected Hilbert space, $\mathcal{H}_{\mathbf{A}}^+$, given by

$$\mathcal{H}_{\mathbf{A}}^+ := P_{\mathbf{A}}^+ \mathcal{H}, \quad P_{\mathbf{A}}^+ := \mathbb{1}_{[0, \infty)}(D_{\mathbf{A}}), \quad P_{\mathbf{A}}^- := \mathbb{1} - P_{\mathbf{A}}^+. \quad (2.9)$$

A priori we define it on the dense domain $P_{\mathbf{A}}^+ \mathcal{D} \subset \mathcal{H}_{\mathbf{A}}^+$,

$$H_{\gamma}^{\text{np}} \varphi^+ := P_{\mathbf{A}}^+ (D_{\mathbf{A}} - \frac{\gamma}{|\mathbf{x}|} + H_{\text{f}}) \varphi^+, \quad \varphi^+ \in P_{\mathbf{A}}^+ \mathcal{D}, \quad (2.10)$$

$$H_V^{\text{np}} \varphi^+ := P_{\mathbf{A}}^+ (D_{\mathbf{A}} + V + H_{\text{f}}) \varphi^+, \quad \varphi^+ \in P_{\mathbf{A}}^+ \mathcal{D}, \quad (2.11)$$

where $V \in L_{\text{loc}}^2(\mathbb{R}^3, \mathbb{R})$ satisfies $\mathcal{D}(D_0) \subset \mathcal{D}(V)$. In the above definitions we have to take care that the right hand sides are actually well-defined as it is, for instance, not obvious that $\frac{\gamma}{|\mathbf{x}|} P_{\mathbf{A}}^+ \mathcal{D} \subset \mathcal{H}$. It follows, however, from the proof of Lemma 3.4(ii) in [37] that $P_{\mathbf{A}}^+$ maps \mathcal{D} into $\mathcal{D}(D_0) \cap \mathcal{D}(H_{\text{f}}^{\nu})$, for every $\nu > 0$, so that the definitions (2.10) and (2.11) make sense.

As soon as we have shown in Section 3 that the quadratic forms of H_V^{PF} and H_V^{np} are semi-bounded below on the dense domains \mathcal{D} and $P_{\mathbf{A}}^+ \mathcal{D}$, respectively, we may extend them to self-adjoint operators by means of a Friedrichs extension. As already mentioned in the introduction there will, however, be critical values for γ above which the quadratic forms are unbounded below in the case of the Coulomb potential. We prove in Section 3 that, for H_{γ}^{PF} , this critical value is not less than

$$\gamma_c^{\text{PF}} := 2/\pi,$$

which is the critical constant in Kato's inequality, $(2/\pi)|\mathbf{x}|^{-1} \leq \sqrt{-\Delta}$. In the case of the no-pair operator we prove the semi-boundedness of the quadratic form of H_{γ}^{np} , for all $\gamma \in [0, \gamma_c^{\text{np}})$, where

$$\gamma_c^{\text{np}} := 2/(2/\pi + \pi/2). \quad (2.12)$$

The instability of both models above the respective critical values for γ is shown in [28, 29] by means of suitable test functions that drive the energy to minus infinity. For the definition of H_{γ}^{np} in the case $\gamma = \gamma_c^{\text{np}}$ see [29].

It has been shown in [13] that the quadratic form associated with the (electronic) Brown-Ravenhall operator,

$$B_\gamma^{\text{el}} := P_0^+ (D_0 - \frac{\gamma}{|\mathbf{x}|}) P_0^+, \quad (2.13)$$

is semi-bounded on $P_0^+ C_0^\infty(\mathbb{R}^3, \mathbb{C}^4)$ if and only if $\gamma \leq \gamma_c^{\text{np}}$. Thus, for $\gamma \leq \gamma_c^{\text{np}}$, it has a self-adjoint Friedrichs extension which we again denote by B_γ^{el} and which actually satisfies Tix' inequality, $B_\gamma^{\text{el}} \geq (1 - \gamma) P_0^+$, $\gamma \in [0, \gamma_c^{\text{np}}]$; see [49]. We exploit the semi-boundedness of B_γ^{el} in the proof of Theorem 3.6 below.

Finally, we introduce a convention used throughout this review: We will frequently use the symbols H_V^\sharp , H_γ^\sharp , γ_c^\sharp , etc. when we treat both the relativistic Pauli-Fierz and no-pair operators at the same time; that is, \sharp is PF or np.

2.4 How to deal with the non-local terms

Although general strategies to prove the existence of ground states have been developed in the framework of non-relativistic QED [4, 6, 17] the application of these ideas to the models discussed in this review poses a variety of new mathematical problems. This is mainly due to the non-locality of the operators $|D_\mathbf{A}|$ and $P_\mathbf{A}^+$ appearing in H_V^\sharp . In this respect the no-pair operator is considerably more difficult to analyze than the semi-relativistic Pauli-Fierz operator since also the projected potential and radiation field energies become non-local. As a consequence a variety of commutator estimates involving $|D_\mathbf{A}|$, $P_\mathbf{A}^+$, H_f , cut-off functions etc. is required for a spectral analysis of H_V^\sharp . Most of these commutator estimates are based on the observations and facts we collect in this subsection. We shall only present one proof in the present subsection in order to illustrate some simple ideas and defer other technical arguments to Section 9. We introduce a general hypothesis on the coupling function which is sometimes used in the sequel:

Hypothesis 2.1 *The map $\mathbb{R}^3 \times (\mathbb{R}^3 \times \mathbb{Z}_2) \ni (\mathbf{x}, k) \mapsto \mathbf{G}_\mathbf{x}(k)$ is measurable such that $\mathbf{x} \mapsto \mathbf{G}_\mathbf{x}(k)$ is continuously differentiable, for almost every k , and*

$$\mathbf{G}_\mathbf{x}(-\mathbf{k}, \lambda) = \overline{\mathbf{G}_\mathbf{x}(\mathbf{k}, \lambda)}, \quad \mathbf{x} \in \mathbb{R}^3, \text{ a.e. } \mathbf{k}, \lambda \in \mathbb{Z}_2. \quad (2.14)$$

There exist $d_{-1}, d_0, d_1, d_2 \in (0, \infty)$ satisfying

$$2 \int \omega(k)^\ell \|\mathbf{G}(k)\|_\infty^2 dk \leq d_\ell^2, \quad 2 \int \frac{\|\nabla_\mathbf{x} \wedge \mathbf{G}(k)\|_\infty^2}{\omega(k)} dk \leq d_1^2, \quad (2.15)$$

where $\|\mathbf{G}(k)\|_\infty := \sup_\mathbf{x} |\mathbf{G}_\mathbf{x}(k)|$, etc.

We remark that, if (2.14) is fulfilled, then $[A^{(j)}(\mathbf{x}), A^{(k)}(\mathbf{y})] = 0$, for all $j, k \in \{1, 2, 3\}$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$. For later reference we also recall the following well-known relative bounds, valid for every $\psi \in \mathcal{D}(H_f^{1/2})$,

$$\|\boldsymbol{\alpha} \cdot a(\mathbf{G}) \psi\|^2 \leq d_{-1}^2 \|H_f^{1/2} \psi\|^2, \quad (2.16)$$

$$\|\boldsymbol{\alpha} \cdot a^\dagger(\mathbf{G}) \psi\|^2 \leq d_{-1}^2 \|H_f^{1/2} \psi\|^2 + d_0^2 \|\psi\|^2. \quad (2.17)$$

In order to cope with the non-locality of $P_{\mathbf{A}}^+$ we write

$$R_{\mathbf{A}}(iy) := (D_{\mathbf{A}} - iy)^{-1}, \quad y \in \mathbb{R},$$

and use the following representation of the sign function of $D_{\mathbf{A}}$ as a strongly convergent principal value (see Lemma VI.5.6 in [23]),

$$S_{\mathbf{A}} \varphi := D_{\mathbf{A}} |D_{\mathbf{A}}|^{-1} \varphi = \lim_{\tau \rightarrow \infty} \int_{-\tau}^{\tau} R_{\mathbf{A}}(iy) \varphi \frac{dy}{\pi}, \quad \varphi \in \mathcal{H}. \quad (2.18)$$

In addition we observe that

$$|D_{\mathbf{A}}| = S_{\mathbf{A}} D_{\mathbf{A}}, \quad P_{\mathbf{A}}^+ = \frac{1}{2} \mathbb{1} + \frac{1}{2} S_{\mathbf{A}}. \quad (2.19)$$

These formulas reduce computations involving $|D_{\mathbf{A}}|$ or $P_{\mathbf{A}}^+$ to computations involving $D_{\mathbf{A}}$ and integrals over its resolvent. To study the exponential localization it is hence useful to recall that, for all $y \in \mathbb{R}$, $a \in [0, 1]$, and $F \in C^\infty(\mathbb{R}_{\mathbf{x}}^3, \mathbb{R})$ having a fixed sign and satisfying $|\nabla F| \leq a$, we have $iy \in \varrho(D_{\mathbf{A}} + i\boldsymbol{\alpha} \cdot \nabla F)$,

$$R_{\mathbf{A}}^F(iy) := e^F R_{\mathbf{A}}(iy) e^{-F} = (D_{\mathbf{A}} + i\boldsymbol{\alpha} \cdot \nabla F - iy)^{-1} \upharpoonright_{\mathcal{D}(e^{-F})}, \quad (2.20)$$

and

$$\|R_{\mathbf{A}}^F(iy)\| \leq \frac{\sqrt{6}}{\sqrt{1+y^2}} \cdot \frac{1}{1-a^2}. \quad (2.21)$$

For classical vector potentials this essentially follows from a computation we learned from [9]; see also [38] where (2.20) and (2.21) are proved in the form stated above. It is, however, clear that the arguments in [38] work for a quantized vector potential, too. Moreover, it is easy to verify that

$$[R_{\mathbf{A}}(iy), \chi e^F] e^{-F} = R_{\mathbf{A}}(iy) i\boldsymbol{\alpha} \cdot (\nabla \chi + \chi \nabla F) R_{\mathbf{A}}^F(iy), \quad (2.22)$$

where $\chi \in C^\infty(\mathbb{R}_{\mathbf{x}}^3, [0, 1])$ is some smooth function of the electron coordinates and F is as above. Finally, we note that

$$\|i\boldsymbol{\alpha} \cdot (\nabla \chi + \chi \nabla F)\| \leq \|\nabla \chi\|_\infty + a, \quad (2.23)$$

since $\|\boldsymbol{\alpha} \cdot \mathbf{v}\| = |\mathbf{v}|$, $\mathbf{v} \in \mathbb{R}^3$, by the Clifford algebra relations, and $|\nabla F| \leq a$. As an example we treat some commutator estimates whose proofs make use of these remarks and a few further useful observations.

Lemma 2.2 *Assume that $\mathbf{G}_{\mathbf{x}}$ fulfills Hypothesis 2.1. Let χ and F be as above, assume additionally that F is bounded, and set $\check{H}_{\mathbf{f}} := H_{\mathbf{f}} + E$, for some sufficiently large $E \geq 1$ (depending on d_1). Let $V \in L_{\text{loc}}^1(\mathbb{R}^3, \mathbb{R})$ be relatively form-bounded with respect to $\sqrt{-\Delta}$. Then, for all $a_0, \kappa \in [0, 1]$, $\nu \geq 0$, and $a \in [0, a_0]$,*

$$\| |D_{\mathbf{A}}|^\kappa [P_{\mathbf{A}}^+, \chi e^F] e^{-F} \| \leq \text{const}(a_0, \kappa) \cdot (a + \|\nabla \chi\|_\infty), \quad (2.24)$$

$$\| \check{H}_{\mathbf{f}}^\nu [P_{\mathbf{A}}^+, \chi e^F] e^{-F} \check{H}_{\mathbf{f}}^{-\nu} \| \leq \text{const}(a_0, \nu) \cdot (a + \|\nabla \chi\|_\infty), \quad (2.25)$$

$$\| |V|^{1/2} [P_{\mathbf{A}}^+, \chi e^F] e^{-F} \check{H}_{\mathbf{f}}^{-1/2} \| \leq \text{const}(a_0, V) \cdot (a + \|\nabla \chi\|_\infty). \quad (2.26)$$

Notice that the a_0 -dependence of the constants originates from the singularity at $a = 1$ in (2.21). Notice also that we may choose $V = |\mathbf{x}|^{-1}$ in (2.26) in view of Kato's inequality.

Before the proof we further remark that all operators appearing in the norms in (2.24)–(2.26) and in similar estimates below are always well-defined a priori on \mathcal{D} and have unique bounded extensions to the whole Hilbert space. In fact, $P_{\mathbf{A}}^+ \mathcal{D} \subset \mathcal{D}(D_{\mathbf{0}}) \cap \bigcap_{\nu > 0} \mathcal{D}(H_{\mathbf{f}}^\nu)$ as we have recalled from [37] above already. To simplify the presentation we shall not comment on this anymore from now on.

PROOF: We use the fact that an operator, T , acting in some Hilbert space is bounded if and only if $\sup_{\|\varphi\|, \|\psi\|=1} |\langle \varphi | T \psi \rangle|$ is bounded in which case it is equal to $\|T\|$. Here it is sufficient to take the supremum over all normalized φ and ψ from a dense set which is a core for T . Combining (2.18), (2.19), and (2.22) we find, for all normalized $\varphi, \psi \in \mathcal{D}$,

$$\begin{aligned} & \langle |D_{\mathbf{A}}|^\kappa \varphi | [P_{\mathbf{A}}^+, \chi e^F] e^{-F} \psi \rangle \\ &= \lim_{\tau \rightarrow \infty} \int_{-\tau}^{\tau} \langle |D_{\mathbf{A}}|^\kappa \varphi | R_{\mathbf{A}}(iy) i\alpha \cdot (\nabla \chi + \chi \nabla F) R_{\mathbf{A}}^F(iy) \psi \rangle \frac{dy}{2\pi}. \end{aligned}$$

On account of $\| |D_{\mathbf{A}}|^\kappa R_{\mathbf{A}}(iy) \| \leq \text{const}(\kappa)(1 + y^2)^{-1/2 + \kappa/2}$, (2.21), and (2.23) we see that the scalar product under the integral sign defines some Lebesgue integrable function of y and

$$|\langle |D_{\mathbf{A}}|^\kappa \varphi | [P_{\mathbf{A}}^+, \chi e^F] e^{-F} \psi \rangle| \leq \text{const}(\kappa) (\|\nabla \chi\|_\infty + a) \int_{\mathbb{R}} \frac{dy}{(1 + y^2)^{1 - \kappa/2}},$$

where the last integral is finite. Therefore, $[P_{\mathbf{A}}^+, \chi e^F] e^{-F} \psi$ belongs to the domain of $(|D_{\mathbf{A}}|^\kappa)^* = |D_{\mathbf{A}}|^\kappa$ and the first bound (2.24) follows.

In order to prove the second bound (2.25) we introduce another little tool which turns out to be useful in our whole analysis. Namely, if $E \geq 1$ is sufficiently large depending on d_1 we can construct $\Upsilon_\nu^F(iy) \in \mathcal{L}(\mathcal{H})$ such that $R_{\mathbf{A}}^F(iy) \check{H}_{\mathbf{f}}^{-\nu} = \check{H}_{\mathbf{f}}^{-\nu} R_{\mathbf{A}}^F(iy) \Upsilon_\nu^F(iy)$, for every $y \in \mathbb{R}$, and such that the norm of $\Upsilon_\nu^F(iy)$ is uniformly bounded with respect to $y \in \mathbb{R}$; see Corollary 9.2 below. (In particular, $R_{\mathbf{A}}^F(iy)$ maps $\mathcal{D}(H_{\mathbf{f}}^\nu)$ into itself.) Therefore,

$$\begin{aligned} & |\langle H_{\mathbf{f}}^\nu \varphi | R_{\mathbf{A}}(iy) i\alpha \cdot (\nabla \chi + \chi \nabla F) R_{\mathbf{A}}^F(iy) \check{H}_{\mathbf{f}}^{-\nu} \psi \rangle| \\ &= |\langle \varphi | R_{\mathbf{A}}(iy) \Upsilon_\nu^0(iy) i\alpha \cdot (\nabla \chi + \chi \nabla F) R_{\mathbf{A}}^F(iy) \Upsilon_\nu^F(iy) \psi \rangle| \\ &\leq C (\|\nabla \chi\|_\infty + a)(1 + y^2)^{-1}, \end{aligned}$$

and it is clear from the argument above how to derive (2.25).

The last bound (2.26) follows from the first two and the inequality $|V|/C \leq |D_{\mathbf{A}}| + H_{\mathbf{f}} + E$ proved later on in Theorem 3.4. \square

3 Self-adjointness

As it is obvious from the definitions in the preceding section the operators H_0^\sharp are positive. In this section we present some basic relative bounds that allow to

define the perturbed operators H_V^\sharp as self-adjoint Friedrichs extensions. As a rule we denote the self-adjoint extensions of H_V^\sharp or H_V^\sharp – which are only defined on \mathcal{D} and $P_A^+ \mathcal{D}$ so far – again by the same symbols. For suitable V , we are also able to characterize the quadratic form domains of H_V^\sharp which turn out to be the spaces of all vectors with finite kinetic and radiation field energy; see Theorems 3.4 and 3.6 below.

Before we present the afore-mentioned results we discuss various (essentially well-known) diamagnetic inequalities in QED; see Theorem 3.2 below. Since these estimates are of independent interest we decided to present one way to derive them (adapted from [47]) in detail which has not been worked out in the literature before, as it seems to us.

3.1 Diamagnetic inequalities in QED

In this subsection it is sufficient to assume that

$$\mathbf{A} = \int_{\mathbb{R}^3}^\oplus \mathbf{A}(\mathbf{x}) d^3 \mathbf{x}, \quad \mathbf{A}(\mathbf{x}) := \mathbb{1}_{\mathbb{C}^4} \otimes (a^\dagger(\mathbf{G}_{\mathbf{x}}) + a(\mathbf{G}_{\mathbf{x}})),$$

where $\mathbb{R}^3 \times (\mathbb{R}^3 \times \mathbb{Z}_2) \ni (\mathbf{x}, k) \mapsto \mathbf{G}_{\mathbf{x}}(k)$ is measurable such that $\mathbf{x} \mapsto \mathbf{G}_{\mathbf{x}}(k)$ is continuously differentiable, for almost every k , and

$$\begin{aligned} \int (\omega(k)^{-1} + \omega(k)^2) \sup_{\mathbf{x}} |\mathbf{G}_{\mathbf{x}}(k)|^2 dk &< \infty, \\ \int (1 + \omega(k)^{-1}) \sup_{\mathbf{x}} |\nabla_{\mathbf{x}} \mathbf{G}_{\mathbf{x}}(k)|^2 dk &< \infty. \end{aligned}$$

The following result is probably well-known but the argument sketched in its proof might be new.

Lemma 3.1 *Let $\lambda \geq 0$. Under the above condition on $\mathbf{G}_{\mathbf{x}}$ the operator $(-i\nabla + \mathbf{A})^2 + \lambda H_f$ is essentially self-adjoint on \mathcal{D} .*

PROOF: It is a standard exercise to show that $\{-i\nabla, \mathbf{A}\} + \mathbf{A}^2$ is a small operator perturbation of $-\Delta + c H_f$, provided that $c > 0$ is chosen sufficiently large depending on $\mathbf{G}_{\mathbf{x}}$. In particular, $\mathcal{N} := (-i\nabla + \mathbf{A})^2 + c H_f + 1$ is essentially self-adjoint on any core of $-\Delta + c H_f$ and, in particular, on \mathcal{D} . In the next step we apply Nelson's commutator theorem with the closure of \mathcal{N} starting from \mathcal{D} as test operator to conclude. \square

We denote the closure of $(-i\nabla + \mathbf{A})^2$ starting from \mathcal{D} by $\tau_{\mathbf{A}}$. For every $\phi, \psi \in \mathcal{H} = L^2(\mathbb{R}_{\mathbf{x}}^3, \mathbb{C}^4 \otimes \mathcal{F}_{\mathbf{b}})$, we write $(\phi | \psi)$ for the (partial) scalar product on $\mathbb{C}^4 \otimes \mathcal{F}_{\mathbf{b}}$ and denote $\llbracket \varphi \rrbracket(\mathbf{x}) := (\varphi(\mathbf{x}) | \varphi(\mathbf{x}))^{1/2}$. Furthermore, we set $S_\phi(\mathbf{x}) := \frac{1}{\llbracket \phi \rrbracket(\mathbf{x})} \phi(\mathbf{x})$, for $\phi(\mathbf{x}) \neq 0$, and $S_\phi(\mathbf{x}) = 0$, for $\phi(\mathbf{x}) = 0$.

Theorem 3.2 (i) *Let $\phi \in \mathcal{D}(\tau_{\mathbf{A}})$. Then $\llbracket \phi \rrbracket \in H^1(\mathbb{R}^3)$, and*

$$\langle \eta | -\Delta \llbracket \phi \rrbracket \rangle_{L^2(\mathbb{R}^3)} \leq \operatorname{Re} \int_{\mathbb{R}^3} \eta(\mathbf{x}) (S_\phi(\mathbf{x}) | (\tau_{\mathbf{A}} \phi)(\mathbf{x})) d^3 \mathbf{x}, \quad (3.1)$$

for all $\eta \in H^1(\mathbb{R}^3)$, $\eta \geq 0$. In particular, for $\eta = \llbracket \phi \rrbracket$,

$$\langle \llbracket \phi \rrbracket | -\Delta \llbracket \phi \rrbracket \rangle_{L^2(\mathbb{R}^3)} \leq \langle \phi | \tau_{\mathbf{A}} \phi \rangle. \quad (3.2)$$

(ii) Let $\phi \in \mathcal{D}(\tau_{\mathbf{A}}^{1/2})$. Then $\llbracket \phi \rrbracket \in H^{1/2}(\mathbb{R}^3)$, and

$$\langle \eta | \sqrt{-\Delta} \llbracket \phi \rrbracket \rangle_{L^2(\mathbb{R}^3)} \leq \operatorname{Re} \int_{\mathbb{R}^3} \eta(\mathbf{x}) (S_{\phi}(\mathbf{x}) | (\tau_{\mathbf{A}}^{1/2} \phi)(\mathbf{x})) d^3 \mathbf{x}, \quad (3.3)$$

for all $\eta \in H^{1/2}(\mathbb{R}^3)$, $\eta \geq 0$. In particular, for $\eta = \llbracket \phi \rrbracket$,

$$\langle \llbracket \phi \rrbracket | \sqrt{-\Delta} \llbracket \phi \rrbracket \rangle_{L^2(\mathbb{R}^3)} \leq \langle \phi | \tau_{\mathbf{A}}^{1/2} \phi \rangle. \quad (3.4)$$

(iii) For all $\psi \in \mathcal{H}$ and $t \in [0, \infty)$, we have, almost everywhere on \mathbb{R}^3 ,

$$\llbracket e^{-t\tau_{\mathbf{A}}} \psi \rrbracket \leq e^{-t(-\Delta)} \llbracket \psi \rrbracket, \quad (3.5)$$

$$\llbracket e^{-t\tau_{\mathbf{A}}^{1/2}} \psi \rrbracket \leq e^{-t\sqrt{-\Delta}} \llbracket \psi \rrbracket. \quad (3.6)$$

Remark 3.3 (1) Arguing as in Theorem 7.21 of [30] with the corresponding changes as in the proof below one can easily verify that, for $\phi \in \mathcal{D}$, we have $\llbracket \phi \rrbracket \in H^1(\mathbb{R}^3)$ and $|\nabla \llbracket \phi \rrbracket(\mathbf{x})| \leq \llbracket (-i\nabla + \mathbf{A})\phi \rrbracket(\mathbf{x})$, for a.e. $\mathbf{x} \in \mathbb{R}^3$.

(2) In [19] diamagnetic inequalities for infra-red regularized vector potentials have been proved by means of dressing transformations. For an alternative proof using functional integrals see [20]. If all components of the vector potential commute, $[A^{(j)}(\mathbf{x}), A^{(k)}(\mathbf{y})] = 0$, then one can also reduce the diamagnetic inequalities to classical ones by diagonalizing all components $A^{(j)}(\mathbf{x})$ simultaneously; this argument due to J. Fröhlich is mentioned in [1]. The proofs given here are variants of the ones presented in [43, 47].

PROOF: Let $\varepsilon > 0$. First, we assume that $\phi \in \mathcal{D}$ and set $u_{\varepsilon} := \sqrt{\llbracket \phi \rrbracket^2 + \varepsilon^2} \in C^{\infty}(\mathbb{R}^3, \mathbb{R})$. Since $A^{(j)}(\mathbf{x})$ is symmetric on \mathcal{C}_0 , for every $\mathbf{x} \in \mathbb{R}^3$, we have $\operatorname{Re}(\phi(\mathbf{x}) | iA^{(j)}(\mathbf{x}) \phi(\mathbf{x})) = 0$, thus

$$u_{\varepsilon} \nabla u_{\varepsilon} = \frac{1}{2} \nabla u_{\varepsilon}^2 = \operatorname{Re}(\phi | \nabla \phi) = \operatorname{Re}(\phi | (\nabla + i\mathbf{A}) \phi). \quad (3.7)$$

In particular,

$$|\nabla u_{\varepsilon}| \leq \frac{\llbracket \phi \rrbracket}{u_{\varepsilon}} \llbracket (\nabla + i\mathbf{A}) \phi \rrbracket \leq \llbracket (\nabla + i\mathbf{A}) \phi \rrbracket. \quad (3.8)$$

Taking the divergence of (3.7) we obtain

$$\begin{aligned} |\nabla u_{\varepsilon}|^2 + u_{\varepsilon} \Delta u_{\varepsilon} &= \operatorname{Re}(\nabla \phi | (\nabla + i\mathbf{A}) \phi) + \operatorname{Re}(\phi | \nabla (\nabla + i\mathbf{A}) \phi) \\ &= \llbracket (\nabla + i\mathbf{A}) \phi \rrbracket^2 - \operatorname{Re}(\phi | \tau_{\mathbf{A}} \phi). \end{aligned} \quad (3.9)$$

Combining this identity with (3.8) we arrive at

$$-\Delta u_{\varepsilon} \leq \operatorname{Re}(u_{\varepsilon}^{-1} \phi | \tau_{\mathbf{A}} \phi). \quad (3.10)$$

Now, assume that $\phi \in \mathcal{D}(\tau_{\mathbf{A}})$. Since $\tau_{\mathbf{A}}$ is essentially self-adjoint on \mathcal{D} we find $\phi_n \in \mathcal{D}$, $n \in \mathbb{N}$, such that $\phi_n \rightarrow \phi$ and $\tau_{\mathbf{A}}\phi_n \rightarrow \tau_{\mathbf{A}}\phi$ in \mathcal{H} . On account of (3.10) we have

$$\int_{\mathbb{R}^3} (-\Delta \eta)(\mathbf{x}) u_{\varepsilon}^{(n)}(\mathbf{x}) d^3 \mathbf{x} \leq \operatorname{Re} \langle \eta (u_{\varepsilon}^{(n)})^{-1} \phi_n | \tau_{\mathbf{A}} \phi_n \rangle, \quad (3.11)$$

for all Schwartz functions $\eta \in \mathcal{S}(\mathbb{R}^3)$, $\eta \geq 0$, where $u_{\varepsilon}^{(n)} := \sqrt{\llbracket \phi_n \rrbracket^2 + \varepsilon^2}$, $n \in \mathbb{N}$. Passing to appropriate subsequences if necessary we may assume that $\llbracket \phi_n \rrbracket \rightarrow \llbracket \phi \rrbracket$ and, hence, $u_{\varepsilon}^{(n)} \rightarrow u_{\varepsilon}$ almost everywhere. Using that $u_{\varepsilon}^{-1}, (u_{\varepsilon}^{(n)})^{-1} \leq 1/\varepsilon$, it is easy to see that $\eta (u_{\varepsilon}^{(n)})^{-1} \phi_n \rightarrow \eta u_{\varepsilon}^{-1} \phi$ in \mathcal{H} . By virtue of the Riesz-Fischer theorem we further find a square-integrable majorant for the sequence $(\llbracket \phi_n \rrbracket)$. We can thus pass to the limit $n \rightarrow \infty$ in (3.11) to get, for all $\eta \in \mathcal{S}(\mathbb{R}^3)$, $\eta \geq 0$, and $\phi \in \mathcal{D}(\tau_{\mathbf{A}})$,

$$\int_{\mathbb{R}^3} (-\Delta \eta)(\mathbf{x}) u_{\varepsilon}(\mathbf{x}) d^3 \mathbf{x} \leq \operatorname{Re} \int_{\mathbb{R}^3} \eta(\mathbf{x}) (u_{\varepsilon}^{-1}(\mathbf{x}) \phi(\mathbf{x}) | (\tau_{\mathbf{A}} \phi)(\mathbf{x})) d^3 \mathbf{x}. \quad (3.12)$$

Here we may take the limit $\varepsilon \rightarrow 0$ by means of the dominated convergence theorem (with the majorant $\eta \llbracket \tau_{\mathbf{A}} \phi \rrbracket$ on the right hand side) to obtain, for all $\eta \in \mathcal{S}(\mathbb{R}^3)$, $\eta \geq 0$, and $\phi \in \mathcal{D}(\tau_{\mathbf{A}})$,

$$\int_{\mathbb{R}^3} (-\Delta \eta)(\mathbf{x}) \llbracket \phi \rrbracket(\mathbf{x}) d^3 \mathbf{x} \leq \operatorname{Re} \int_{\mathbb{R}^3} \eta(\mathbf{x}) (S_{\phi}(\mathbf{x}) | (\tau_{\mathbf{A}} \phi)(\mathbf{x})) d^3 \mathbf{x}. \quad (3.13)$$

Adding $\int \eta \lambda \llbracket \phi \rrbracket$ with $\lambda > 0$ to both sides we obtain

$$\begin{aligned} \int_{\mathbb{R}^3} [(-\Delta + \lambda) \eta](\mathbf{x}) \llbracket \phi \rrbracket(\mathbf{x}) d^3 \mathbf{x} &\leq \operatorname{Re} \int_{\mathbb{R}^3} \eta(\mathbf{x}) (S_{\phi} | (\tau_{\mathbf{A}} + \lambda) \phi)(\mathbf{x}) d^3 \mathbf{x} \\ &\leq \int_{\mathbb{R}^3} \eta(\mathbf{x}) \llbracket (\tau_{\mathbf{A}} + \lambda) \phi \rrbracket(\mathbf{x}) d^3 \mathbf{x}. \end{aligned}$$

Let $0 \leq \chi \in C_0^{\infty}(\mathbb{R}^3)$ and $\psi \in \mathcal{H}$. Since $(-\Delta + \lambda)^{-1}$, $\lambda > 0$, is positivity preserving, we may then choose $\eta := (-\Delta + \lambda)^{-1} \chi \in \mathcal{S}(\mathbb{R}^3)$ and $\phi := (\tau_{\mathbf{A}} + \lambda)^{-1} \psi \in \mathcal{D}(\tau_{\mathbf{A}})$ and arrive at

$$\int_{\mathbb{R}^3} \chi(\mathbf{x}) \llbracket (\tau_{\mathbf{A}} + \lambda)^{-1} \psi \rrbracket(\mathbf{x}) d^3 \mathbf{x} \leq \int_{\mathbb{R}^3} \chi(\mathbf{x}) [(-\Delta + \lambda)^{-1} \llbracket \psi \rrbracket](\mathbf{x}) d^3 \mathbf{x}.$$

Since $\chi \in C_0^{\infty}(\mathbb{R}^3)$ is arbitrary we find $\llbracket (\tau_{\mathbf{A}} + \lambda)^{-1} \psi \rrbracket \leq (-\Delta + \lambda)^{-1} \llbracket \psi \rrbracket$, almost everywhere on \mathbb{R}^3 , and by induction (see [47], for the same argument) we get, for all $n \in \mathbb{N}$ and $t > 0$,

$$\llbracket \left(\frac{n}{t}\right)^n (\tau_{\mathbf{A}} + \frac{n}{t})^{-n} \psi \rrbracket \leq \left(\frac{n}{t}\right)^n (-\Delta + \frac{n}{t})^{-n} \llbracket \psi \rrbracket.$$

Both sides converge almost everywhere along some subsequence to $\llbracket e^{-t\tau_{\mathbf{A}}} \psi \rrbracket$ and $e^{-t(-\Delta)} \llbracket \psi \rrbracket$ respectively, and (3.5) follows. Equation (3.6) follows from (3.5), the spectral calculus, the subordination identity

$$e^{-t\lambda^{1/2}} = \int_0^{\infty} e^{-s-t^2\lambda/(4s)} \frac{ds}{\sqrt{\pi s}}, \quad t, \lambda \geq 0,$$

and the properties of the Bochner-Lebesgue integral. In the remaining part of the proof we derive (following again [47]) (3.1) and (3.3) at the same time. To this end let $\nu \in \{1/2, 1\}$ and $\phi \in \mathcal{D}(\tau_{\mathbf{A}}^\nu)$.

On account of $\langle \phi | e^{-t\tau_{\mathbf{A}}^\nu} \phi \rangle \leq \int \llbracket \phi \rrbracket \llbracket e^{-t\tau_{\mathbf{A}}^\nu} \phi \rrbracket$ Equations (3.5) and (3.6) imply $\langle \phi | e^{-t\tau_{\mathbf{A}}^\nu} \phi \rangle \leq \int \llbracket \phi \rrbracket e^{-t(-\Delta)^\nu} \llbracket \phi \rrbracket$, thus

$$\langle \phi | t^{-1}(1 - e^{-t\tau_{\mathbf{A}}^\nu}) \phi \rangle \geq \int_{\mathbb{R}^3} t^{-1}(1 - e^{-t|\xi|^{2\nu}}) |\llbracket \phi \rrbracket|^2(\xi) d^3\xi.$$

Since $\phi \in \mathcal{D}(\tau_{\mathbf{A}}^\nu)$ the limit $t \rightarrow 0$ exists on the left hand side of the previous inequality. By the monotone convergence theorem we conclude that the limit $t \rightarrow 0$ of the right hand side exists, too, and

$$\langle \phi | \tau_{\mathbf{A}}^\nu \phi \rangle \geq \int_{\mathbb{R}^3} |\xi|^{2\nu} |\llbracket \phi \rrbracket|^2(\xi) d^3\xi.$$

Hence $\llbracket \phi \rrbracket \in H^\nu(\mathbb{R}^3)$ and (3.2) and (3.4) hold true. Using this we may take the derivatives at $t = 0$ on the left and right sides of the following consequence of (3.5) and (3.6),

$$\begin{aligned} \operatorname{Re} \int_{\mathbb{R}^3} \eta(\mathbf{x}) (S_\phi | e^{-t\tau_{\mathbf{A}}^\nu} \phi)(\mathbf{x}) d^3\mathbf{x} &\leq \int_{\mathbb{R}^3} \eta(\mathbf{x}) \llbracket e^{-t\tau_{\mathbf{A}}^\nu} \phi \rrbracket(\mathbf{x}) d^3\mathbf{x} \\ &\leq \int_{\mathbb{R}^3} \eta(\mathbf{x}) e^{-t(-\Delta)^\nu} \llbracket \phi \rrbracket(\mathbf{x}) d^3\mathbf{x}, \end{aligned} \quad (3.14)$$

to get $-\operatorname{Re} \langle \eta S_\phi | \tau_{\mathbf{A}}^\nu \phi \rangle \leq -\langle \eta | (-\Delta)^\nu \llbracket \phi \rrbracket \rangle_{L^2(\mathbb{R}^3)}$, for all $\eta \in H^\nu(\mathbb{R}^3)$, $\eta \geq 0$. Here we have also used that all expressions in (3.14) are equal to $\int \eta \llbracket \phi \rrbracket$ at $t = 0$. \square

3.2 Semi-boundedness

The following theorem is a slight generalization of a result from [37]. Its proof is based on two basic steps: The first one follows immediately from the diamagnetic inequalities by means of which the form bounds the potential satisfies by assumption can be turned into form bounds with respect to the *scalar* operators $\tau_{\mathbf{A}}^{1/2}$ or $\tau_{\mathbf{A}}$. After that we use relative bounds on the magnetic field to include spin, $\tau_{\mathbf{A}} \leq D_{\mathbf{A}}^2 + H_{\mathbf{f}}^2 + \text{const}$. To complete the square on the right hand side of the previous bound we employ the inequality (3.20) below. After completing the square we take square roots on both sides to obtain a bound on $\tau_{\mathbf{A}}^{1/2}$.

Theorem 3.4 *For $\nu \in \{1/2, 1\}$, let $V_\nu \in L_{\text{loc}}^1(\mathbb{R}^3, \mathbb{R})$ and assume that there is some $c_\nu \geq 0$ such that, for every $\varphi \in H^\nu(\mathbb{R}^3)$,*

$$\langle \varphi | V_\nu \varphi \rangle_{L^2(\mathbb{R}^3)} \leq \langle \varphi | (-\Delta)^\nu \varphi \rangle_{L^2(\mathbb{R}^3)} + c_\nu \|\varphi\|_{L^2(\mathbb{R}^3)}^2. \quad (3.15)$$

Then there is some $C \in (0, \infty)$ such that, for all $\mathbf{G}_{\mathbf{x}}$ fulfilling Hypothesis 2.1, $\phi \in \mathcal{D}$, and $\delta > 0$,

$$\langle \phi | V_\nu \phi \rangle \leq \langle \phi | (|D_{\mathbf{A}}| + \delta H_{\mathbf{f}} + (\delta^{-1} + C\delta) d_1^2)^{2\nu} \phi \rangle + c_\nu \|\phi\|^2. \quad (3.16)$$

In particular,

$$\frac{1}{4} \left\| |\mathbf{x}|^{-1} \phi \right\|^2 \leq \left\| (|D_{\mathbf{A}}| + \delta H_f + (\delta^{-1} + C \delta) d_1^2) \phi \right\|^2, \quad (3.17)$$

for all $\phi \in \mathcal{D}$, and

$$\frac{2}{\pi} \frac{1}{|\mathbf{x}|} \leq |D_{\mathbf{A}}| + \delta H_f + (\delta^{-1} + C \delta) d_1^2, \quad (3.18)$$

in the sense of quadratic forms on \mathcal{D} . Therefore, $H_{V_{1/2}}^{\text{PF}}$ and H_{γ}^{PF} , $\gamma \in [0, \gamma_c^{\text{PF}}]$, have self-adjoint Friedrichs extensions – henceforth again denoted by the same symbols – and \mathcal{D} is a form core for these extensions. Moreover, for $a \in [0, 1)$ and $\gamma \in [0, \gamma_c^{\text{PF}})$, we know that $\mathcal{Q}(H_{aV_{1/2}}^{\text{PF}}) = \mathcal{Q}(H_{\gamma}^{\text{PF}}) = \mathcal{Q}(H_0^{\text{PF}}) = \mathcal{Q}(|D_0|) \cap \mathcal{Q}(H_f)$.

PROOF: First, we show that $\mathcal{Q}(H_0^{\text{PF}}) = \mathcal{Q}(|D_0|) \cap \mathcal{Q}(H_f)$; see [28, 48]. The remaining statements on form domains will then be a consequence of (3.16) and (3.18). In fact, this follows from the bounds [37]

$$\left\| |D_0|^{1/2} (S_{\mathbf{A}} - S_0) \check{H}_f^{-1/2} \right\| \leq C, \quad \left\| |D_{\mathbf{A}}|^{1/2} (S_{\mathbf{A}} - S_0) \check{H}_f^{-1/2} \right\| \leq C,$$

where $\check{H}_f := H_f + E$ with $E \geq 1 + (4d_1)^2$. (These bounds are derived exactly as in Lemma 6.3 below.) Together with $|D_{\mathbf{A}}| - |D_0| = D_0 (S_{\mathbf{A}} - S_0) + \boldsymbol{\alpha} \cdot \mathbf{A} S_{\mathbf{A}}$ and (2.16)&(2.17) the first bound implies

$$\begin{aligned} \left| \langle \varphi | (|D_{\mathbf{A}}| - |D_0|) \varphi \rangle \right| &\leq C' \left\| |D_0|^{1/2} \varphi \right\| \left\| \check{H}_f^{1/2} \varphi \right\| \\ &\leq C'' \langle \varphi | (|D_0| + H_f) \varphi \rangle, \end{aligned} \quad (3.19)$$

for all $\varphi \in \mathcal{D}$. Analogously, the second bound implies (3.19) with $|D_0|$ replaced by $|D_{\mathbf{A}}|$ on the right hand side. Consequently, the form norms of $|D_{\mathbf{A}}| + H_f$ and $|D_0| + H_f$ are equivalent on \mathcal{D} which implies $\mathcal{Q}(H_0^{\text{PF}}) = \mathcal{Q}(|D_0|) \cap \mathcal{Q}(H_f)$.

All details missing in the proof of (3.16)–(3.18) sketched below can be found in [37]. We set $\check{H}_f := H_f + E$, for $E > 0$. Besides some standard arguments the main ingredient in this proof is the following bound proven in [37, Lemma 4.1]: We find some constant, $C > 0$, such that, for all $E > C d_1^2$ and $\phi \in \mathcal{D}$,

$$\text{Re} \langle |D_{\mathbf{A}}| \phi | \check{H}_f \phi \rangle \geq 0. \quad (3.20)$$

This estimate follows from the following identity $\text{Re}(|D_{\mathbf{A}}| \check{H}_f) = \check{H}_f^{1/2} (|D_{\mathbf{A}}| - \mathcal{T}) \check{H}_f^{1/2}$ on \mathcal{D} , where

$$\mathcal{T} := \text{Re} \{ [|D_{\mathbf{A}}|, \check{H}_f^{-1/2}] \check{H}_f^{1/2} \} \leq \varepsilon |D_{\mathbf{A}}| + \varepsilon^{-1} \text{const } d_1 / E^{1/2},$$

for $\varepsilon \in (0, 1]$ and $E \geq (4d_1)^2$, as we shall see at the end of Subsection 9.2. To make use of the bound (3.20) we recall that, since $[A^{(j)}(\mathbf{x}), A^{(k)}(\mathbf{y})] = 0$, we have

$$D_{\mathbf{A}}^2 \phi = \tau_{\mathbf{A}} \phi + \mathbf{S} \cdot \mathbf{B} \phi + \phi, \quad \phi \in \mathcal{D}, \quad (3.21)$$

where the entries of the formal three-vector \mathbf{S} are $S_j = \sigma_j \otimes \mathbb{1}_2$ and \mathbf{B} is the magnetic field, i.e. $\mathbf{S} \cdot \mathbf{B} = \mathbf{S} \cdot a^\dagger(\nabla_{\mathbf{x}} \wedge \mathbf{G}) + \mathbf{S} \cdot a(\nabla_{\mathbf{x}} \wedge \mathbf{G})$. By (2.16) with $(\nabla_{\mathbf{x}} \wedge \mathbf{G}, d_1)$ instead of (\mathbf{G}, d_{-1}) we have, for all $\delta > 0$ and $\phi \in \mathcal{D}$,

$$|\langle \phi | \mathbf{S} \cdot \mathbf{B} \phi \rangle| \leq 2 d_1 \|\phi\| \|H_f^{1/2} \phi\| \leq \delta \langle \phi | (H_f + \delta^{-2} d_1^2) \phi \rangle. \quad (3.22)$$

Choosing $E = (\delta^{-2} + C) d_1^2$ we infer from (3.20)–(3.22), for all $\phi \in \mathcal{D}$,

$$\begin{aligned} \langle \phi | \tau_{\mathbf{A}} \phi \rangle &\leq \langle D_{\mathbf{A}} \phi | D_{\mathbf{A}} \phi \rangle + \delta \langle \phi | \check{H}_f \phi \rangle - \|\phi\|^2 \\ &\leq \langle D_{\mathbf{A}} \phi | D_{\mathbf{A}} \phi \rangle + \langle \phi | \delta^2 \check{H}_f^2 \phi \rangle + 2\text{Re} \langle |D_{\mathbf{A}}| \phi | \delta \check{H}_f \phi \rangle \\ &= \|(|D_{\mathbf{A}}| + \delta \check{H}_f) \phi\|^2. \end{aligned} \quad (3.23)$$

Furthermore, since the square root is operator monotone it follows from (3.23) that $\langle \phi | \tau_{\mathbf{A}}^{1/2} \phi \rangle \leq \langle \phi | (|D_{\mathbf{A}}| + \delta \check{H}_f) \phi \rangle$. Using the diamagnetic inequalities (3.2) and (3.4) we further find, for $\nu \in \{1/2, 1\}$,

$$\begin{aligned} \langle \phi | V_\nu \phi \rangle &= \langle \llbracket \phi \rrbracket | V_\nu \llbracket \phi \rrbracket \rangle_{L^2(\mathbb{R}^3)} \leq \langle \llbracket \phi \rrbracket | (-\Delta)^\nu \llbracket \phi \rrbracket \rangle_{L^2(\mathbb{R}^3)} + c_\nu \|\phi\|^2 \\ &\leq \langle \phi | \tau_{\mathbf{A}}^{2\nu} \phi \rangle + c_\nu \|\phi\|^2, \end{aligned} \quad (3.24)$$

and we conclude that (3.16) holds true. Inequalities (3.17) and (3.18) follow from (3.24) together with Hardy's and Kato's inequality, respectively. \square

Theorem 3.4 has a straightforward extension to the case of N electrons [36]. We discuss this extension in the next corollary mainly since its proof gives the opportunity to introduce some identities and estimates which are used later on. Let \mathcal{H}_N and \mathcal{D}_N , $N \in \mathbb{N}$, be defined in the same way as \mathcal{H} and \mathcal{D} but with the $L^2(\mathbb{R}^3, \mathbb{C}^4) = L^2(\mathbb{R}^3 \times \mathbb{Z}_4)$ replaced by $L^2((\mathbb{R}^3 \times \mathbb{Z}_4)^N)$. The spatial coordinates of the i -th electron are denoted by $\mathbf{x}_i \in \mathbb{R}^3$ and we designate an operator acting only on \mathbf{x}_i , the i -th spinor components, and on the photon field by a superscript (i) .

Corollary 3.5 *Assume that $N, K \in \mathbb{N}$, $e > 0$, $\gamma_1, \dots, \gamma_K \in (0, 2/\pi]$, and $\{\mathbf{R}_1, \dots, \mathbf{R}_K\} \subset \mathbb{R}^3$. Then*

$$\sum_{i=1}^N |D_{\mathbf{A}}^{(i)}| - \sum_{i=1}^N \sum_{k=1}^K \frac{\gamma_k}{|\mathbf{x}_i - \mathbf{R}_k|} + \sum_{i < j} \frac{e^2}{|\mathbf{x}_i - \mathbf{x}_j|} + \delta H_f > -\infty, \quad (3.25)$$

for every $\delta > 0$, in the sense of quadratic forms on \mathcal{D}_N .

PROOF: In view of (3.18) we only have to explain how to localize the non-local kinetic energy terms. To begin with we note the following special cases of (2.24) and (9.10), respectively: For every $\chi \in C^\infty(\mathbb{R}_{\mathbf{x}}^3, [0, 1])$,

$$\|\chi, S_{\mathbf{A}}\| \leq \|\nabla \chi\|_\infty, \quad \|D_{\mathbf{A}} [\chi, [\chi, S_{\mathbf{A}}]]\| \leq 2 \|\nabla \chi\|_\infty^2. \quad (3.26)$$

Let $\mathcal{B}_r(\mathbf{z})$ denote the open ball in \mathbb{R}^3 of radius $r > 0$ centered at $\mathbf{z} \in \mathbb{R}^3$. We set $\varrho := \min\{|\mathbf{R}_k - \mathbf{R}_\ell| : k \neq \ell\}/2$ and pick a smooth partition of unity

on \mathbb{R}^3 , $\{\chi_k\}_{k=0}^K$, such that $\chi_k \equiv 1$ on $\mathcal{B}_{\varrho/2}(\mathbf{R}_k)$ and $\text{supp}(\chi_k) \subset \mathcal{B}_{\varrho}(\mathbf{R}_k)$, for $k = 1, \dots, K$, and such that $\sum_{k=0}^K \chi_k^2 = 1$. Combining the following IMS type localization formula,

$$|D_{\mathbf{A}}| = \sum_{k=0}^K \left\{ \chi_k |D_{\mathbf{A}}| \chi_k + \frac{1}{2} [\chi_k, [\chi_k, |D_{\mathbf{A}}|]] \right\} \quad \text{on } \mathcal{D}, \quad (3.27)$$

and the identities

$$[\chi_k, [\chi_k, |D_{\mathbf{A}}|]] = 2i\boldsymbol{\alpha} \cdot (\nabla \chi_k) [\chi_k, S_{\mathbf{A}}] + D_{\mathbf{A}} [\chi_k, [\chi_k, S_{\mathbf{A}}]] \quad (3.28)$$

and $\|\boldsymbol{\alpha} \cdot \nabla \chi_k\| = |\nabla \chi_k|$ with (3.26), we obtain

$$\|[\chi_k, [\chi_k, |D_{\mathbf{A}}|]]\| \leq 4 \|\nabla \chi_k\|_{\infty}^2, \quad (3.29)$$

for all $k \in \{0, \dots, K\}$. Since we are able to localize the kinetic energy terms and since, by the choice of the partition of unity, the functions $\mathbb{R}^3 \ni \mathbf{x} \mapsto |\mathbf{x} - \mathbf{R}_k|^{-1} \chi_k^2(\mathbf{x})$ are bounded, for $k \in \{1, \dots, K\}$, $\ell \in \{0, \dots, K\}$, $k \neq \ell$, the bound (3.25) is now an immediate consequence of (3.18). \square

Next, we discuss the semi-boundedness of the no-pair operator. The idea is to reduce the stability of the no-pair operator to the one of the purely electronic Brown-Ravenhall operator.

Theorem 3.6 *Assume that $\mathbf{G}_{\mathbf{x}}$ fulfills Hypothesis 2.1. Let $\delta \in (0, 1]$ and let $V \in L_{\text{loc}}^2(\mathbb{R}^3, \mathbb{R})$ be form bounded with respect to $\sqrt{-\Delta}$ and satisfy*

$$\langle P_0^+ \varphi | V P_0^+ \varphi \rangle \leq a \langle \varphi | D_0 P_0^+ \varphi \rangle + b \|P_0^+ \varphi\|^2, \quad \varphi \in C_0^\infty(\mathbb{R}^3), \quad (3.30)$$

for some $a \in (0, 1)$ and $b \geq 0$. Then there exist constants $c_V, C \in (0, \infty)$, $C \equiv C(\delta, V, d_{-1}, d_1)$, such that, for all $\varphi^+ \in P_{\mathbf{A}}^+ \mathcal{D}$, $\|\varphi^+\| = 1$,

$$\langle \varphi^+ | (D_{\mathbf{A}} + V + \delta H_{\text{f}}) \varphi^+ \rangle \geq c_V \langle \varphi^+ | |D_0| \varphi^+ \rangle - C, \quad (3.31)$$

and in particular, for every $\gamma \in [0, \gamma_c^{\text{np}})$,

$$\langle \varphi^+ | (D_{\mathbf{A}} - \frac{\gamma}{|\mathbf{x}|} + \delta H_{\text{f}}) \varphi^+ \rangle \geq c_\gamma \langle \varphi^+ | |D_0| \varphi^+ \rangle - C(\delta, \gamma, d_{-1}, d_1). \quad (3.32)$$

Therefore, H_V^{np} and H_γ^{np} have self-adjoint Friedrichs extensions – henceforth again denoted by the same symbols – and $P_{\mathbf{A}}^+ \mathcal{D}$ is a form core for these extensions. Furthermore, $\mathcal{Q}(H_V^{\text{np}}) = \mathcal{Q}(H_\gamma^{\text{np}}) = \mathcal{Q}(H_0^{\text{np}}) = \mathcal{Q}(|D_0|) \cap \mathcal{Q}(H_{\text{f}}) \cap \text{Ran } P_{\mathbf{A}}^+$.

PROOF: The statement on the form domains follows from Theorem 3.2 of [29]. (See also Section 3.4 of the *first* preprint version of [29] available on the arXiv for an alternative proof.) The estimate (3.32) is derived in [37] and we shall outline its proof in what follows.

We pick some $\rho > 1$ with $\rho a < 1$ and write, for $\varphi^+ \in P_{\mathbf{A}}^+ \mathcal{D}$,

$$\begin{aligned} \langle \varphi^+ | (D_{\mathbf{A}} + V) \varphi^+ \rangle &= \rho^{-1} \langle \varphi^+ | P_{\mathbf{0}}^+ (D_{\mathbf{0}} + \rho V) P_{\mathbf{0}}^+ \varphi^+ \rangle \\ &\quad + (1 - \rho^{-1}) \langle \varphi^+ | P_{\mathbf{0}}^+ D_{\mathbf{0}} \varphi^+ \rangle \\ &\quad + \langle \varphi^+ | \boldsymbol{\alpha} \cdot \mathbf{A} \varphi^+ \rangle \\ &\quad + \langle \varphi^+ | P_{\mathbf{0}}^- (D_{\mathbf{0}} + V) P_{\mathbf{0}}^- \varphi^+ \rangle \\ &\quad + 2 \operatorname{Re} \langle \varphi^+ | P_{\mathbf{0}}^+ V P_{\mathbf{0}}^- \varphi^+ \rangle. \end{aligned} \quad (3.33)$$

The estimate (3.32) is based on the identity above and the following bound on the difference between the spectral projections with and without field (see Lemma 6.3 for similar statements): For $E \geq 1 + (4d_1)^2$, there is a constant, $C \equiv C(d_{-1}, d_0) > 0$, such that

$$\| |D_{\mathbf{0}}|^{3/4} (P_{\mathbf{0}}^{\pm} - P_{\mathbf{A}}^{\pm}) \check{H}_{\mathbf{f}}^{-1/2} \| \leq C, \quad (3.34)$$

where $\check{H}_{\mathbf{f}} = H_{\mathbf{f}} + E$. On account of $P_{\mathbf{0}}^- \varphi^+ = (P_{\mathbf{0}}^- - P_{\mathbf{A}}^-) \varphi^+$, for $\varphi^+ \in P_{\mathbf{A}}^+ \mathcal{D}$, and (3.34) we have, for every $\varepsilon > 0$,

$$\begin{aligned} \| |D_{\mathbf{0}}|^{1/2} P_{\mathbf{0}}^- \varphi^+ \|^2 &\leq \| |D_{\mathbf{0}}|^{1/4} P_{\mathbf{0}}^- \varphi^+ \| \| |D_{\mathbf{0}}|^{3/4} (P_{\mathbf{0}}^- - P_{\mathbf{A}}^-) \varphi^+ \| \\ &\leq \frac{1}{2} \| |D_{\mathbf{0}}|^{1/2} P_{\mathbf{0}}^- \varphi^+ \|^2 + \frac{C(\varepsilon, d_{-1}, d_0)}{2} \| P_{\mathbf{0}}^- \varphi^+ \|^2 + \frac{\varepsilon}{2} \| \check{H}_{\mathbf{f}}^{1/2} \varphi^+ \|^2, \end{aligned}$$

that is,

$$\| |D_{\mathbf{0}}|^{1/2} P_{\mathbf{0}}^- \varphi^+ \|^2 \leq \varepsilon \| \check{H}_{\mathbf{f}}^{1/2} \varphi^+ \|^2 + C(\varepsilon, d_{-1}, d_0) \| P_{\mathbf{0}}^- \varphi^+ \|^2. \quad (3.35)$$

By virtue of $|V| \leq C |D_{\mathbf{0}}|$, the previous estimate further implies, for every $\tau > 0$,

$$\begin{aligned} &| \langle P_{\mathbf{0}}^+ \varphi^+ | V P_{\mathbf{0}}^- \varphi^+ \rangle | \\ &\leq \tau \| |D_{\mathbf{0}}|^{1/2} P_{\mathbf{0}}^+ \varphi^+ \|^2 + \varepsilon \| \check{H}_{\mathbf{f}}^{1/2} \varphi^+ \|^2 + C_{\varepsilon, \tau} \| P_{\mathbf{0}}^- \varphi^+ \|^2. \end{aligned} \quad (3.36)$$

Here the second term on the RHS of (3.33) can be used to control the first term on the RHS of (3.36). Recalling the definition (2.13) and applying (2.16), (3.35), (3.36), and (3.30) to the various terms in (3.33) we thus find, for every $\delta \in (0, 1]$, some constant, $C \equiv C(\delta, \rho, d_{-1}, d_1) \in (0, \infty)$, such that

$$\langle \varphi^+ | (D_{\mathbf{A}} + V + \delta H_{\mathbf{f}}) \varphi^+ \rangle \geq c_{a, \rho} \langle \varphi^+ | D_{\mathbf{0}} P_{\mathbf{0}}^+ \varphi^+ \rangle - C \| \varphi^+ \|^2.$$

Using (3.35) once more to replace $D_{\mathbf{0}} P_{\mathbf{0}}^+$ by $|D_{\mathbf{0}}|$ on the right hand side, we arrive at the first asserted estimate. According to the remarks made below (2.13) the first estimate applies in particular to the Coulomb potential, as long as $\gamma \in [0, \gamma_c^{\text{np}})$. \square

From the previous theorem and our commutator estimates one can also infer the semi-boundedness of a no-pair operator for $N \in \mathbb{N}$ electrons and $K \in \mathbb{N}$ static nuclei, analogously to Corollary 3.5, as long as all Coulomb coupling constants $\gamma_1, \dots, \gamma_K$ are less than γ_c^{np} ; see Proposition A.2 of [36].

Since we are addressing the question of finding distinguished self-adjoint realizations of H_{γ}^{\sharp} it is also natural to state the following theorem whose proof can be found in Corollary 3.4 of [29].

Theorem 3.7 *Let $\gamma \in [0, 1/2)$ and assume that $\mathbf{G}_\mathbf{x}$ fulfills Hypothesis 2.1. Then H_γ^{PF} and H_γ^{np} are essentially self-adjoint on \mathcal{D} and $P_\Lambda^+ \mathcal{D}$, respectively.*

For sufficiently small values of $|e|$ and/or Λ , the essential self-adjointness of H_0^{PF} has been shown earlier in [40].

4 Bounds on the ionization energy

As a first step towards the proof of the existence of ground states we need to show that binding occurs in the atomic system defined by H_V^\sharp in the sense that $\inf \sigma[H_V^\sharp] < \inf \sigma[H_0^\sharp]$. This information will be exploited mathematically when we apply a bound on the spatial localization of low-lying spectral subspaces of H_V^\sharp from [37]. The localization estimate in turn enters into the proof of the existence of ground states at various places, for instance, into the derivation of the infra-red estimates and into the compactness argument given in Subsection 8.1. Theorem 4.1 below is the main result of this section. In its statement we abbreviate ($\sharp \in \{\text{PF}, \text{np}\}$)

$$E_V^\sharp := \inf \sigma[H_V^\sharp], \quad E_\gamma^\sharp := \inf \sigma[H_\gamma^\sharp], \quad \gamma \in (0, \gamma_c^\sharp), \quad \Sigma^\sharp := \inf \sigma[H_0^\sharp],$$

where V satisfies the conditions under which H_V^\sharp has been defined in the previous section. To simplify the exposition we only consider the physical choice of the coupling function $\mathbf{G}_\mathbf{x}$ given in (2.2), as always for arbitrary values of e and Λ . Our proofs work, however, equally well for other coupling functions, for instance, for the infra-red cut-off and discretized coupling functions introduced in Section 6, and we obtain uniform bounds on the binding energies in these cases. If we consider coupling functions other than (2.2) then the unitary transformation U introduced below has to be changed accordingly; see [28, 29].

Theorem 4.1 (i) *Let $V \in L_{\text{loc}}^2(\mathbb{R}^3, \mathbb{R})$ be form bounded with respect to $\sqrt{-\Delta}$ with form bound less than or equal to one. (So V fulfills (3.15) with $\nu = 1/2$.) Define the self-adjoint operator $h_V := \sqrt{1 - \Delta} + V$ by means of a Friedrichs extension starting from $C_0^\infty(\mathbb{R}^3)$ and assume that $\inf \sigma[h_V]$ is an eigenvalue. Then*

$$\Sigma^{\text{PF}} - E_V^{\text{PF}} \geq 1 - \inf \sigma[h_V]. \quad (4.1)$$

In particular, for $\gamma \in [0, \gamma_c^{\text{PF}}]$,

$$\Sigma^{\text{PF}} - E_\gamma^{\text{PF}} \geq 1 - \inf \sigma[\sqrt{1 - \Delta} - \frac{\gamma}{|\mathbf{x}|}]. \quad (4.2)$$

(ii) *Let $V \leq 0$ be relatively form bounded with respect to $\sqrt{-\Delta}$ and assume that V satisfies (3.30) with $a \in (0, 1)$. Additionally, assume there exist $r \geq 1$, $c > 0$, and $\theta \in (0, 2)$ such that*

$$V(\mathbf{x}) \leq -c |\mathbf{x}|^{\theta-2}, \quad |\mathbf{x}| \geq r.$$

Then

$$\Sigma^{\text{np}} - E_V^{\text{np}} > 0, \quad (4.3)$$

and in particular, for $\gamma \in (0, \gamma_c^{\text{np}})$,

$$\Sigma^{\text{np}} - E_\gamma^{\text{np}} > 0.$$

Remark 4.2 (i) The bound (4.1) has been obtained first in [21] (under the assumption that H_V^{PF} be essentially self-adjoint which, in the case $V = -\frac{\gamma}{|\mathbf{x}|}$, is true, at least for all $\gamma < 1/2$). The result of [21] improved a lower bound on the binding energy in an earlier preprint version of [28]. The latter was given in terms of the *non-relativistic* ground state energy of an electronic Schrödinger operator.

(ii) In a forthcoming work of the first two authors [27] it is shown that the inequalities (4.1) and (4.2) are actually *strict*, for all $e, \Lambda > 0$. Moreover, there is a certain class of short-range potentials V such that $\Sigma^{\text{PF}} - E_V^{\text{PF}} > 0$ and in particular – according to the present article – E_V^{PF} is an eigenvalue of H_V^{PF} although $\inf \sigma[h_V] = 1$ and 1 is not an eigenvalue of h_V . This effect is called *enhanced binding* due to the quantized radiation field and we are again able to prove its occurrence, for arbitrary large values of e and Λ . There are numerous results on enhanced binding in non-relativistic QED; up to now complete proofs were, however, available only for small e . The proofs in [27] extend the ideas and methods underlying the proof of Theorem 4.1 given below.

Because of lack of space we shall only describe the proof of Theorem 4.1 for the semi-relativistic Pauli-Fierz operator [28] in detail. The proof of (4.3) follows similar lines and can be found in [29]; see also Remark (4.3) below.

Our proof of (4.1) and (4.3) is based on a direct fiber decomposition of \mathcal{H} with respect to fixed values of the total momentum $\mathbf{p} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbf{p}_f$, where $\mathbf{p} := -i\nabla_{\mathbf{x}}$ and

$$\mathbf{p}_f := d\Gamma(\mathbf{k}) := (d\Gamma(k^{(1)}), d\Gamma(k^{(2)}), d\Gamma(k^{(3)})) \quad (4.4)$$

is the photon momentum operator. In fact, a conjugation of the Dirac operator with the unitary operator $e^{i\mathbf{p}_f \cdot \mathbf{x}}$ – which is simply a multiplication with the phase $(\mathbb{R}^3)^n \ni (\mathbf{k}_1, \dots, \mathbf{k}_n) \mapsto e^{i(\mathbf{k}_1 + \dots + \mathbf{k}_n) \cdot \mathbf{x}}$ in each Fock space sector $\mathcal{F}_b^{(n)}[\mathcal{K}]$ – yields

$$e^{i\mathbf{p}_f \cdot \mathbf{x}} D_{\mathbf{A}} e^{-i\mathbf{p}_f \cdot \mathbf{x}} = \boldsymbol{\alpha} \cdot (\mathbf{p} - \mathbf{p}_f + \mathbf{A}(\mathbf{0})) + \beta,$$

and a further conjugation with the Fourier transform, $\mathcal{F} : L^2(\mathbb{R}_{\mathbf{x}}^3) \rightarrow L^2(\mathbb{R}_{\mathbf{P}}^3)$, turns the latter expressions into

$$(\mathcal{F} \otimes \mathbb{1}) e^{i\mathbf{p}_f \cdot \mathbf{x}} D_{\mathbf{A}} e^{-i\mathbf{p}_f \cdot \mathbf{x}} (\mathcal{F}^{-1} \otimes \mathbb{1}) = \int_{\mathbb{R}^3}^{\oplus} \widehat{D}(\mathbf{P}) d^3\mathbf{P}. \quad (4.5)$$

Here the operators

$$\widehat{D}(\mathbf{P}) := \boldsymbol{\alpha} \cdot (\mathbf{P} - \mathbf{p}_f + \mathbf{A}(\mathbf{0})) + \beta, \quad \mathbf{P} \in \mathbb{R}^3,$$

acting in $\mathbb{C}^4 \otimes \mathcal{F}_b[\mathcal{K}]$, are fiber Hamiltonians of the transformed Dirac operator in (4.5) with respect to the isomorphism

$$\mathcal{H} = L^2(\mathbb{R}_{\mathbf{P}}^3, \mathbb{C}^4) \otimes \mathcal{F}_b[\mathcal{K}] \cong \int_{\mathbb{R}^3}^{\oplus} \mathbb{C}^4 \otimes \mathcal{F}_b[\mathcal{K}] d^3\mathbf{P}. \quad (4.6)$$

(In particular, the transformed Dirac operator in (4.5) again acts in \mathcal{H} , where the variable in the first tensor factor, \mathbf{P} , is now interpreted as the total momentum of the combined electron-photon system.) Accordingly, we have the direct integral representation (compare, e.g., Theorem XIII.85 in [44])

$$(\mathcal{F} \otimes \mathbb{1}) e^{i\mathbf{p}_f \cdot \mathbf{x}} H_0^{\text{PF}} e^{-i\mathbf{p}_f \cdot \mathbf{x}} (\mathcal{F}^{-1} \otimes \mathbb{1}) = \int_{\mathbb{R}^3}^{\oplus} H_0^{\text{PF}}(\mathbf{P}) d^3\mathbf{P}, \quad (4.7)$$

where

$$H_0^{\text{PF}}(\mathbf{P}) := |\widehat{D}(\mathbf{P})| + H_f.$$

Let $\varepsilon > 0$. Then we know that the Lebesgue measure of the set of all $\mathbf{P} \in \mathbb{R}^3$ satisfying $\sigma[H_0^{\text{PF}}(\mathbf{P})] \cap (\Sigma^{\text{PF}} - \varepsilon, \Sigma^{\text{PF}} + \varepsilon) \neq \emptyset$ is strictly positive. In particular, we find some $\mathbf{P}_\star \in \mathbb{R}^3$ and some normalized $\varphi_\star \in \mathcal{Q}(H_0^{\text{PF}}(\mathbf{P}_\star))$ such that

$$\langle \varphi_\star | H_0^{\text{PF}}(\mathbf{P}_\star) \varphi_\star \rangle_{\mathbb{C}^4 \otimes \mathcal{F}_b[\mathcal{H}]} < \Sigma^{\text{PF}} + \varepsilon. \quad (4.8)$$

We define the unitary transformation

$$U := e^{i(\mathbf{p}_f - \mathbf{P}_\star) \cdot \mathbf{x}} \quad (4.9)$$

and observe as above that

$$U D_{\mathbf{A}} U^* = \widehat{D}_{\mathbf{p}}(\mathbf{P}_\star) := \boldsymbol{\alpha} \cdot (\mathbf{p} + \mathbf{P}_\star - \mathbf{p}_f + \mathbf{A}(\mathbf{0})) + \beta.$$

It is sufficient to prove the bound (4.1) for the unitarily equivalent operator

$$U H_V^{\text{PF}} U^* = |\widehat{D}_{\mathbf{p}}(\mathbf{P}_\star)| + V + H_f. \quad (4.10)$$

PROOF: (*Theorem 4.1: The semi-relativistic Pauli-Fierz case.*) Let $\varepsilon > 0$ and \mathbf{P}_\star be as in the paragraph preceding the statement. We abbreviate $\mathbf{t}_\star := \mathbf{P}_\star - \mathbf{p}_f + \mathbf{A}(\mathbf{0})$ and, for $\eta \geq 0$,

$$R_1(\eta) := (\mathbf{p}^2 + (\boldsymbol{\alpha} \cdot \mathbf{t}_\star)^2 + \eta + 1)^{-1}, \quad R_2(\eta) := ((\boldsymbol{\alpha} \cdot (\mathbf{p} + \mathbf{t}_\star))^2 + \eta + 1)^{-1}.$$

Since the anti-commutator of $\boldsymbol{\alpha} \cdot \mathbf{p}$ and $\boldsymbol{\alpha} \cdot \mathbf{t}_\star$ is equal to $2\mathbf{p} \cdot \mathbf{t}_\star$ it holds $(\boldsymbol{\alpha} \cdot (\mathbf{p} + \mathbf{t}_\star))^2 = (\boldsymbol{\alpha} \cdot \mathbf{t}_\star)^2 + 2\mathbf{p} \cdot \mathbf{t}_\star + \mathbf{p}^2$. We deduce that, for any $\varphi \in \mathcal{D}$,

$$\begin{aligned} -R_2(\eta)\varphi &= -R_2(\eta) [\mathbf{p}^2 + (\boldsymbol{\alpha} \cdot \mathbf{t}_\star)^2 + 1 + \eta] R_1(\eta)\varphi \\ &= R_2(\eta) [2\mathbf{p} \cdot \mathbf{t}_\star] R_1(\eta)\varphi - R_1(\eta)\varphi. \end{aligned} \quad (4.11)$$

We use the following formula, for a self-adjoint operator $T > 0$,

$$\sqrt{T}\varphi = \int_0^\infty \left(1 - \frac{\eta}{T + \eta}\right) \varphi \frac{d\eta}{\pi\sqrt{\eta}}, \quad \varphi \in \mathcal{D}(T), \quad (4.12)$$

and the resolvent identity (4.11) to obtain, for any $\varphi \in \mathcal{D}$,

$$\begin{aligned}
& \langle \varphi | (\sqrt{(\boldsymbol{\alpha} \cdot (\mathbf{p} + \mathbf{t}_*))^2 + 1} - \sqrt{\mathbf{p}^2 + (\boldsymbol{\alpha} \cdot \mathbf{t}_*)^2 + 1}) \varphi \rangle \\
&= \int_0^\infty \langle \varphi | (R_1(\eta) - R_2(\eta)) \varphi \rangle \sqrt{\eta} \frac{d\eta}{\pi} \\
&= \int_0^\infty \langle R_2(\eta) \varphi | [2\mathbf{p} \cdot \mathbf{t}_*] R_1(\eta) \varphi \rangle \sqrt{\eta} \frac{d\eta}{\pi} \\
&= \int_0^\infty \langle \varphi | R_1(\eta) [2\mathbf{p} \cdot \mathbf{t}_*] R_1(\eta) \varphi \rangle \sqrt{\eta} \frac{d\eta}{\pi} \\
&\quad - \int_0^\infty \langle \varphi | R_1(\eta) [2\mathbf{p} \cdot \mathbf{t}_*] R_2(\eta) [2\mathbf{p} \cdot \mathbf{t}_*] R_1(\eta) \varphi \rangle \sqrt{\eta} \frac{d\eta}{\pi} \\
&\leq \int_0^\infty \langle \varphi | R_1(\eta) [2\mathbf{p} \cdot \mathbf{t}_*] R_1(\eta) \varphi \rangle \sqrt{\eta} \frac{d\eta}{\pi}. \tag{4.13}
\end{aligned}$$

In the last step we used the positivity of $R_2(\eta)$. We consider now $\varphi := \varphi_1 \otimes \varphi_2$ where $\varphi_1 \in C_0^\infty(\mathbb{R}^3, \mathbb{R})$ and $\varphi_2 \in \mathbb{C}^4 \otimes \mathcal{C}_0$ with $\|\varphi_j\| = 1$, $j = 1, 2$. Writing $\Phi_2(\boldsymbol{\xi}, \eta) := (\boldsymbol{\xi}^2 + (\boldsymbol{\alpha} \cdot \mathbf{t}_*)^2 + 1 + \eta)^{-1} \varphi_2 = \Phi_2(-\boldsymbol{\xi}, \eta)$ we find that

$$\begin{aligned}
& \langle \varphi | R_1(\eta) \mathbf{p} \cdot \mathbf{t}_* R_1(\eta) \varphi \rangle \\
&= \int_{\mathbb{R}^3} \boldsymbol{\xi} \cdot \langle \Phi_2(\boldsymbol{\xi}, \eta) | \mathbf{t}_* \Phi_2(\boldsymbol{\xi}, \eta) \rangle |\widehat{\varphi}_1(\boldsymbol{\xi})|^2 d^3\boldsymbol{\xi} = 0, \tag{4.14}
\end{aligned}$$

due to the fact that φ_1 is real and, hence, $|\widehat{\varphi}_1(\boldsymbol{\xi})| = |\widehat{\varphi}_1(-\boldsymbol{\xi})|$. Furthermore,

$$|\widehat{D}_{\mathbf{p}}(\mathbf{P}_*)| = \sqrt{(\boldsymbol{\alpha} \cdot (\mathbf{p} + \mathbf{t}_*))^2 + 1}, \quad |\widehat{D}(\mathbf{P}_*)| = \sqrt{(\boldsymbol{\alpha} \cdot \mathbf{t}_*)^2 + 1}, \tag{4.15}$$

$$\sqrt{\mathbf{p}^2 + (\boldsymbol{\alpha} \cdot \mathbf{t}_*)^2 + 1} \leq \sqrt{(\boldsymbol{\alpha} \cdot \mathbf{t}_*)^2 + 1} + \sqrt{\mathbf{p}^2 + 1} - 1, \tag{4.16}$$

where we used $[\mathbf{p}^2, (\boldsymbol{\alpha} \cdot \mathbf{t}_*)^2] = 0$ in the second line. Combining (4.13)–(4.16) we arrive at

$$\langle \varphi | UH_V^{\text{PF}} U^* \varphi \rangle \leq \langle \varphi_2 | H_0^{\text{PF}}(\mathbf{P}_*) \varphi_2 \rangle + \langle \varphi_1 | h_V \varphi_1 \rangle - 1.$$

By a limiting argument the previous inequality extends to any real-valued $\varphi_1 \in \mathcal{Q}(h_V)$. We choose φ_1 to be the normalized, strictly positive eigenfunction of h_V corresponding to the eigenvalue at the bottom of its spectrum and $\varphi_2 = \varphi_*$. By the choice of φ_* in (4.8), where $\varepsilon > 0$ is arbitrary, this proves the assertion. \square

Remark 4.3 As already mentioned the proof of Theorem 4.1 for the no-pair operator employs ideas similar to those described above. However, due to the more complex structure of the no-pair Hamiltonian the resulting bound (4.3) is not as satisfactory as the one for H_V^{PF} . Again, we have the representation $H_0^{\text{np}} \cong \int_{\mathbb{R}^3}^\oplus H_0^{\text{np}}(\mathbf{P}) d^3\mathbf{P}$ with

$$H_0^{\text{np}}(\mathbf{P}) = \widehat{P}(\mathbf{P}) (\widehat{D}(\mathbf{P}) + H_{\text{f}}) \widehat{P}(\mathbf{P}), \quad \widehat{P}(\mathbf{P}) := \mathbb{1}_{[0, \infty)}(\widehat{D}(\mathbf{P})).$$

Given $\varepsilon > 0$, we again find $\mathbf{P}_\star \in \mathbb{R}^3$ and some normalized $\varphi_\star = \widehat{P}(\mathbf{P}) \varphi_\star \in \mathcal{Q}(H^{\text{np}}(\mathbf{P}_\star))$ such that $\langle \varphi_\star | H_0^{\text{np}}(\mathbf{P}_\star) \varphi_\star \rangle_{\mathbb{C}^4 \otimes \mathcal{F}_b[\mathcal{H}]} < \Sigma^{\text{np}} + \varepsilon$. Conjugating H_γ^{np} with U defined as in (4.9) we obtain

$$U H_\gamma^{\text{np}} U^* = \widehat{P}_\mathbf{p}(\mathbf{P}_\star) (\widehat{D}_\mathbf{p}(\mathbf{P}_\star) + V H_f) \widehat{P}_\mathbf{p}(\mathbf{P}_\star), \quad (4.17)$$

where $\widehat{P}_\mathbf{p}(\mathbf{P}_\star) := \mathbb{1}_{[0, \infty)}(\widehat{D}_\mathbf{p}(\mathbf{P}_\star))$. As a test function we now choose $\Phi_R := \widehat{P}_\mathbf{p}(\mathbf{P}_\star)(\chi_R \otimes \varphi_\star)$, where χ_R is some normalized smooth and non-negative function supported in $\{R \leq |\mathbf{x}| \leq 2R\}$ with $R \geq 1$. It turns out that $\|\Phi_R\| = 1 + \mathcal{O}(1/R^2)$. We then exploit the fact that the negative contribution of the potential in (4.17) to the expectation value of Φ_R decays as $1/R^{2-\vartheta}$ whereas all other terms yield a contribution $\Sigma^{\text{np}} + \varepsilon + \mathcal{O}(1/R^2)$, as R tends to infinity. To show all this it is convenient to work on a non-projected Hilbert space by adding a suitable “positronic” no-pair operator to H_γ^{np} (similarly as in (5.1) below). For then one can again make use of the bounds (4.13)–(4.16) derived in the previous proof; see Section V of [29]. (It is entirely obvious that the Coulomb potential can be replaced by a more general one in Section V of [29].)

5 Exponential localization

Our next aim is to discuss the exponential localization with respect to the electron coordinates of low-lying spectral subspaces of the no-pair and semi-relativistic Pauli-Fierz operators. Since the multiplication with some exponential weight function, e^F , acting on the electron coordinates does not map the projected Hilbert space $\mathcal{H}_\mathbf{A}^+$ into itself it is convenient to extend the no-pair operator to some continuously invertible operator on the whole Hilbert space \mathcal{H} in the discussion below. Therefore, we set

$$\widehat{H}_V^{\text{np}} := H_V^{\text{np}} + H_0^{\text{np}, -}, \quad H_0^{\text{np}, -} := P_\mathbf{A}^- (|D_\mathbf{A}| + H_f) P_\mathbf{A}^-. \quad (5.1)$$

In Lemma 8.3 below we show that $\Sigma^{\text{np}} = \inf \sigma[H_0^{\text{np}, -}]$ by constructing some anti-linear map $\tau : \mathcal{H} \rightarrow \mathcal{H}$ with $H_0^{\text{np}} \tau = \tau H_0^{\text{np}, -}$. Therefore,

$$\widehat{H}_0^{\text{np}} \geq \Sigma^{\text{np}}. \quad (5.2)$$

To unify the notation we further set $\widehat{H}_V^{\text{PF}} := H_V^{\text{PF}}$ and write \widehat{H}_V^\sharp , Σ^\sharp , etc., when we treat both H_V^{PF} and $\widehat{H}_V^{\text{np}}$ at the same time. In the whole section we assume that $\mathbf{G}_\mathbf{x}$ fulfills Hypothesis 2.1.

Theorem 5.1 (i) *Let $V \in L_{\text{loc}}^2(\mathbb{R}^3, \mathbb{R})$ be relatively form bounded with respect to $\sqrt{-\Delta}$ with relative form bound less than or equal to one. Moreover, assume that*

$$\exists r \geq 1 : \quad \sup_{|\mathbf{x}| \geq r/4} |V(\mathbf{x})| < \infty \quad \text{and} \quad V(\mathbf{x}) \xrightarrow{|\mathbf{x}| \rightarrow \infty} 0. \quad (5.3)$$

Define

$$\rho(a) := 1 - (1 - a^2)^{1/2}, \quad a \in [0, 1],$$

and let $I \subset (-\infty, \Sigma^{\text{PF}})$ be some compact interval. Then there exists $k \in (0, \infty)$, such that, for all $a \in (0, 1)$ satisfying $\varepsilon := \Sigma^{\text{PF}} - \sup I - \rho(a) \in (0, 1]$, we have

$$\|e^{a|\mathbf{x}|} \mathbb{1}_I(H_V^{\text{PF}})\| \leq k (\Sigma^{\text{PF}} - E_V^{\text{PF}}) e^{k/\varepsilon}. \quad (5.4)$$

(ii) Assume that $V \in L_{\text{loc}}^2(\mathbb{R}^3, \mathbb{R})$ satisfies $H^1(\mathbb{R}^3) \subset \mathcal{D}(V)$ (which implies $|V| \leq \text{const} |D_0|$) as well as (3.30) with $a < 1$ and (5.3). Let $I \subset (-\infty, \Sigma^{\text{np}})$ be some compact interval. Then we find some $a' > 0$ such that $\text{Ran}(\mathbb{1}_I(\hat{H}_V^{\text{np}})) \subset \mathcal{D}(e^{a'|\mathbf{x}|})$. In particular,

$$\|e^{a'|\mathbf{x}|} \mathbb{1}_I(H_V^{\text{np}})\|_{\mathcal{L}(\mathcal{H}_A^+, \mathcal{H})} \leq \text{const}. \quad (5.5)$$

If $\mathbf{G}_{\mathbf{x}}$ is modified, then we get uniform lower bounds on a' and uniform upper bounds on the constant in (5.5), provided that we have uniform upper bounds on $d_{-1}, d_1, \Sigma^{\text{np}}$ and uniform lower bounds on $\Sigma^{\text{np}} - \sup I$.

PROOF: The proof is given in the succeeding three subsections. \square
Note that the potential V is *not* assumed to be a *small* form perturbation of $\sqrt{-\Delta}$ in Part (i) of the previous theorem. In particular, the assumptions of (i) cover the Coulomb potential $-\gamma/|\mathbf{x}|$ with coupling constants $\gamma \in [0, \gamma_c^{\text{PF}}]$ including the critical one. This improves on [37] where Coulomb potentials have been treated, for subcritical γ . By a modification of the arguments of this section it is actually also possible to prove exponential localization for the no-pair operator with Coulomb potential in the critical case $\gamma = \gamma_c^{\text{np}}$, which is not covered by Part (ii) of the above theorem; see [25].

The bound on the decay rate a of Part (i) has been found first in [25] (where only the Coulomb potential is treated explicitly). It reduces to the typical relativistic decay rate known for eigenvectors of electronic Dirac or square-root operators when $\mathbf{G}_{\mathbf{x}}$ is set equal to zero.

5.1 A general strategy to prove the localization of spectral subspaces

The general strategy of the proof of Theorem 5.1 is essentially due to [4]. We shall present a variant of the argument used in [4] in Lemma 5.2 below. In order to apply this lemma to H_V^\sharp , we shall also benefit from some useful observations made in [16]. The main advantage of Lemma 5.2 and its earlier variants is that it allows to study the localization of *spectral subspaces* without any a priori knowledge on the spectrum. Its proof does not exploit eigenvalue equations as it is the case in Agmon type arguments nor does it assume discreteness of the spectrum or the presence of spectral gaps. This is important for us since the spectra of both the no-pair and the semi-relativistic Pauli-Fierz operators will be continuous up to their minima. Roughly speaking the proof of Lemma 5.2 rests on a combination of the following:

- The representation (5.11) for the spectral projection $\mathbb{1}_I(\hat{H}_\gamma^\sharp)$. Here a *comparison operator*, Y , enters into the analysis whose resolvent stays bounded

after conjugation with suitable exponential weights, for all relevant spectral parameters. (5.11) is valid since it also satisfies $\mathbb{1}_I(Y) = 0$.

- The *Helfffer-Sjöstrand formula* (re-derived below for the convenience of the reader) which is used to represent smoothed versions of $\mathbb{1}_I(\hat{H}_\gamma^\sharp)$ and $\mathbb{1}_I(Y)$ as integrals over resolvents.
- The second resolvent identity; in fact, Y will be chosen such that $\hat{H}_\gamma^\sharp - Y$ is well-localized and is hence able to control exponential weights.

In the somewhat technical parts of this section following after Lemma 5.2 we shall verify the applicability of Lemma 5.2 to our models by defining and analyzing suitable comparison operators Y .

Let us now introduce some prerequisites for the proof of Lemma 5.2. In order to find a representation of the spectral projection which is accessible for an analysis we smooth out the projection and employ the Helfffer-Sjöstrand formula. More precisely, let $I \subset (-\infty, \Sigma^\sharp)$ be some compact interval. Then we pick some slightly larger compact interval $J \subset (-\infty, \Sigma^\sharp)$, $J \supset I$, and some $\chi \in C_0^\infty(\mathbb{R}, [0, 1])$ such that $\chi \equiv 1$ on I and $\chi \equiv 0$ outside J . We pick another cut-off function $\rho \in C_0^\infty(\mathbb{R}, [0, 1])$ such that $\rho = 1$ in a neighborhood of 0 and $\rho(y) = 0$, for $|y| \geq 1/2$, and extend χ to a compactly supported smooth function of the complex plane setting

$$\tilde{\chi}(x + iy) := \chi(x) + \chi'(x) iy \rho(y), \quad x, y \in \mathbb{R};$$

compare, e.g., [11, 12]. Then we have

$$2 \partial_{\bar{z}} \tilde{\chi}(z) := (\partial_x + i \partial_y) \chi(z) = \chi'(x) (1 - \rho(y) - y \rho'(y)) + \chi''(x) iy \rho(y),$$

for every $z = x + iy \in \mathbb{C}$, and the choice of ρ implies

$$|\partial_{\bar{z}} \tilde{\chi}(z)| \leq C_\chi |\operatorname{Im} z|, \quad z \in \mathbb{C}, \quad (5.6)$$

for some $C_\chi \in (0, \infty)$. Moreover, the following Helfffer-Sjöstrand formula is valid, for every self-adjoint operator, X , on some Hilbert space,

$$\chi(X) = \int_{\mathbb{C}} (X - z)^{-1} d\mu(z), \quad d\mu(z) := -\frac{1}{\pi} \partial_{\bar{z}} \tilde{\chi}(z) dx dy. \quad (5.7)$$

If X were a complex number (5.7) were just a special case of Pompeiu's formula for some path encircling the support of χ . X can, however, be inserted in that formula by means of the spectral calculus; see, e.g., [12].

To facilitate the discussion of operator domains we replace $a|\mathbf{x}|$ in (5.4) by some $F : \mathbb{R}_{\mathbf{x}}^3 \rightarrow \mathbb{R}$ satisfying

$$F \in C^\infty \cap L^\infty(\mathbb{R}_{\mathbf{x}}^3, \mathbb{R}), \quad F = ar \text{ on } \mathcal{B}_{r/2}(0), \quad F \geq 0, \quad |\nabla F| \leq a, \quad (5.8)$$

where $r \geq 1$ is the parameter appearing in (5.3).

Finally, we recall our notation $\sharp \in \{\text{np}, \text{PF}\}$.

Lemma 5.2 *Let I , J , $\tilde{\chi}$, and C_χ be as described above and assume that Y is a self-adjoint operator in \mathcal{H} with $\mathcal{D}(Y) = \mathcal{D}(\hat{H}_V^\sharp)$ and $Y > \sup J$. Furthermore, let $a > 0$ and assume there exist $C, C' \in (0, \infty)$ such that, for all F satisfying (5.8),*

$$\|e^F(\hat{H}_V^\sharp - Y)\| \leq C, \quad \sup_{z \in J + i\mathbb{R}} \|e^F(Y - z)^{-1}e^{-F}\| \leq C'. \quad (5.9)$$

Then $\text{Ran}(\mathbb{1}_I(\hat{H}_V^\sharp)) \subset \mathcal{D}(e^{a|\mathbf{x}|})$ and

$$\|e^{a|\mathbf{x}|} \mathbb{1}_I(\hat{H}_V^\sharp)\| \leq C C' C_\chi \mathcal{L}(\text{supp}(\chi'))/\pi, \quad (5.10)$$

where \mathcal{L} denotes the Lebesgue measure on \mathbb{R} .

PROOF: Since $\mathbb{1}_I = \chi \mathbb{1}_I$ and $J \subset \varrho(Y)$ we have

$$\mathbb{1}_I(\hat{H}_V^\sharp) = (\chi(\hat{H}_V^\sharp) - \chi(Y)) \mathbb{1}_I(\hat{H}_V^\sharp). \quad (5.11)$$

Applying the Helffer-Sjöstrand formula (5.7) and the second resolvent identity we infer that

$$e^F \mathbb{1}_I(\hat{H}_V^\sharp) = \int_{\mathbb{C}} \{e^F(Y - z)^{-1}e^{-F}\} \{e^F(Y - \hat{H}_V^\sharp)\} (\hat{H}_V^\sharp - z)^{-1} d\mu(z). \quad (5.12)$$

Estimating the norm of these expressions using (5.6) and $\|(\hat{H}_V^\sharp - z)^{-1}\| \leq 1/|\text{Im } z|$ we obtain (5.10) with $a|\mathbf{x}|$ replaced by F . Then (5.10) follows by choosing a suitable sequence of functions F_n satisfying (5.8) and converging monotonically on $\{|\mathbf{x}| \geq 2r\}$ to $a|\mathbf{x}|$ and applying the monotone convergence theorem to $e^{F_n} \mathbb{1}_I(\hat{H}_V^\sharp) \psi$, for every $\psi \in \mathcal{H}$. \square

5.2 Choice of the comparison operator Y

In the next step we thus have to find a suitable operator Y fulfilling the conditions of Lemma 5.2. The hardest problem is to verify the second bound in (5.9) and this is actually the main new mathematical challenge in the study of the exponential localization in our non-local models. We defer the discussion of the second bound in (5.9) to the next subsection.

In order to construct an operator Y whose spectrum differs only slightly from the spectrum of the free operator \hat{H}_0^\sharp , so that $Y > \sup J$, and which is defined on the same domain as \hat{H}_V^\sharp we simply add some bounded term to \hat{H}_V^\sharp which compensates for the singularities and wells of the electrostatic potential and thus pushes the spectrum up to the ionization threshold. A similar choice of a comparison operator has been employed in the non-relativistic setting in [16]. More precisely, we first introduce a scaled partition of unity. That is, we pick $\chi_{0,R}, \chi_{1,R} \in C^\infty(\mathbb{R}^3, [0, 1])$, $R \geq r$, such that $\chi_{0,R} \equiv 1$ on $\mathcal{B}_R(0)$, $\chi_{0,R} \equiv 0$ on $\mathbb{R}^3 \setminus \mathcal{B}_{2R}(0)$, $\chi_{0,R}^2 + \chi_{1,R}^2 = 1$, and $\|\nabla \chi_{k,R}\|_\infty \leq c/R$, $k = 0, 1$, for some R -independent constant $c \in (0, \infty)$. Then we define Y as follows:

The semi-relativistic Pauli-Fierz operator. We define

$$Y_V^{\text{PF}} := H_V^{\text{PF}} + (\Sigma^{\text{PF}} - E_V^{\text{PF}}) \chi_{0,R}^2, \quad (5.13)$$

for some $R \geq 1$ which shall be fixed sufficiently large later on. Of course, H_V^{PF} and Y_V^{PF} have the same domain and the first bound in (5.9) which provides the control on the exponential weights holds trivially,

$$\|e^F(H_V^{\text{PF}} - Y_V^{\text{PF}})\| \leq (\Sigma^{\text{PF}} - E_V^{\text{PF}}) \|e^F \chi_{0,R}^2\|_\infty \leq (\Sigma^{\text{PF}} - E_V^{\text{PF}}) e^{ar+2aR},$$

for every F satisfying (5.8). We shall sketch the proof of the condition $Y > \sup J$ which follows from the next lemma.

Lemma 5.3 $Y_V^{\text{PF}} \geq \Sigma^{\text{PF}} - o(R^0)$, $R \rightarrow \infty$, in the sense of quadratic forms on \mathcal{D} , where the little- o -symbol depends only on V and $\chi_{0,1}$.

PROOF: We employ the localization formula (3.27) with $K = 1$, the error estimate (3.29), and $H_V^{\text{PF}} \geq E_V^{\text{PF}}$, $H_0^{\text{PF}} \geq \Sigma^{\text{PF}}$, to get

$$\begin{aligned} Y_V^{\text{PF}} &\geq \chi_{0,R} H_V^{\text{PF}} \chi_{0,R} + \chi_{1,R} H_0^{\text{PF}} \chi_{1,R} \\ &\quad + (\Sigma^{\text{PF}} - E_V^{\text{PF}}) \chi_{0,R}^2 + \chi_{1,R}^2 V - \mathcal{O}(1/R^2) \geq \Sigma^{\text{PF}} - o(R^0). \end{aligned}$$

We also used that $\sup_{|\mathbf{x}| \leq R} |V(\mathbf{x})| = o(R^0)$ and $\chi_{0,R}^2 + \chi_{1,R}^2 = 1$. \square

The no-pair operator. Since $|D_{\mathbf{A}}| = P_{\mathbf{A}}^+ |D_{\mathbf{A}}| + P_{\mathbf{A}}^- |D_{\mathbf{A}}|$ we have

$$\widehat{H}_V^{\text{np}} = |D_{\mathbf{A}}| + P_{\mathbf{A}}^+ V P_{\mathbf{A}}^+ + H_{\text{f}}^{\text{diag}}, \quad H_{\text{f}}^{\text{diag}} := P_{\mathbf{A}}^+ H_{\text{f}} P_{\mathbf{A}}^+ + P_{\mathbf{A}}^- H_{\text{f}} P_{\mathbf{A}}^-,$$

on \mathcal{D} . We write $\widehat{H}_{V,R}^{\text{np}} := \widehat{H}_V^{\text{np}} - (1/R) H_{\text{f}}^{\text{diag}}$, for some $R > 1$, so that $E_{V,R}^{\text{np}} := \inf \sigma[\widehat{H}_{V,R}^{\text{np}} P_{\mathbf{A}}^+] > -\infty$ by (3.31), and define

$$Y_V^{\text{np}} = \widehat{H}_V^{\text{np}} + (\Sigma^{\text{np}} - E_{V,R}^{\text{np}}) \chi_{0,R} P_{\mathbf{A}}^+ \chi_{0,R}. \quad (5.14)$$

Again it is clear that $\widehat{H}_V^{\text{np}}$ and Y_V^{np} are self-adjoint on the same domain. Also the first bound in (5.9) again follows trivially,

$$\|e^F(\widehat{H}_V^{\text{np}} - Y_V^{\text{np}})\| \leq (\Sigma^{\text{np}} - E_{V,R}^{\text{np}}) e^{ar+2aR},$$

for every F satisfying (5.8). Besides the second bound in (5.9) it remains to derive the following lemma.

Lemma 5.4 $Y_V^{\text{np}} \geq \Sigma^{\text{np}} - \Sigma^{\text{np}}/R - o(R^0) - k d_1^2/R^2$ as quadratic forms on \mathcal{D} , where $k \in (0, \infty)$ and the Little- o -symbol depend only on V and $\chi_{0,1}$.

PROOF: Again we use an IMS type localization formula to infer that

$$\begin{aligned} Y_V^{\text{np}} &\geq \chi_{0,R} \widehat{H}_{V,R}^{\text{np}} \chi_{0,R} + \chi_{1,R} \widehat{H}_{0,R}^{\text{np}} \chi_{1,R} + (\Sigma^{\text{np}} - E_{V,R}^{\text{np}}) \chi_{0,R} P_{\mathbf{A}}^+ \chi_{0,R} \\ &\quad + \frac{1}{R} H_{\text{f}}^{\text{diag}} + \chi_{1,R} P_{\mathbf{A}}^+ V P_{\mathbf{A}}^+ \chi_{1,R} + \frac{1}{2} \sum_{k=0,1} [\chi_{k,R}, [\chi_{k,R}, \widehat{H}_{V,R}^{\text{np}}]]. \end{aligned} \quad (5.15)$$

As a consequence of (2.25), (2.26), (3.29), (9.11), (9.12), and $H_f \leq 2H_f^{\text{diag}}$ the double commutator in the last line is bounded from below by $-(k/R^2)(H_f^{\text{diag}} + d_1^2 + 1)$, where $k \in (0, \infty)$ depends only on V and $\chi_{0,1}$. To control this error we use the term $\frac{1}{R} H_f^{\text{diag}}$ in (5.15). Furthermore, we put $\mu_R := \chi_{0,R/2}$, so that $\chi_{1,R} \mu_R = 0$. Then $(1 - \mu_R^2) V = o(R^0)$ and, by (2.26),

$$\begin{aligned} -\chi_{1,R} P_{\mathbf{A}}^+ V P_{\mathbf{A}}^+ \chi_{1,R} &\leq -\chi_{1,R} [P_{\mathbf{A}}^+, \mu_R] V [\mu_R, P_{\mathbf{A}}^+] \chi_{1,R} + o(R^0) \chi_{1,R}^2 \\ &\leq o(R^0) \chi_{1,R}^2 + \mathcal{O}(1/R^2) (H_f^{\text{diag}} + d_1^2 + 1), \end{aligned}$$

so that the second term in the last line of (5.15) can again be controlled by the first one. Using these remarks, $\hat{H}_{V,R}^{\text{np}} \geq E_{V,R}^{\text{np}} P_{\mathbf{A}}^+ + (1 - 1/R) \Sigma^{\text{np}} P_{\mathbf{A}}^-$, and $\hat{H}_{0,R}^{\text{np}} \geq (1 - 1/R) \Sigma^{\text{np}}$ (by (5.2)), we arrive at the assertion. \square

5.3 Conjugation of Y with exponential weights

In order to prove Theorem 5.1 it only remains to verify the second bound in (5.9). The following lemma [37] gives a criterion for this condition to hold true.

Lemma 5.5 *Let Y be a non-negative operator in \mathcal{H} which admits \mathcal{D} as a form core. Set $b := \inf \sigma(Y)$ and let $J \subset (-\infty, b)$ be some compact interval. Let $a \in (0, 1)$ and assume that, for all F satisfying (5.8), we have $e^{\pm F} \mathcal{Q}(Y) \subset \mathcal{Q}(Y)$. (Notice that $e^{\pm F}$ maps \mathcal{D} into itself.) Assume further that there exist constants $c(a), f(a), g(a), h(a) \in [0, \infty)$ such that $c(a) < 1/2$ and $\delta := b - \max J - b g(a) - h(a) > 0$ and, for all F satisfying (5.8) and $\varphi \in \mathcal{D}$,*

$$|\langle \varphi | (e^F Y e^{-F} - Y) \varphi \rangle| \leq c(a) \langle \varphi | Y \varphi \rangle + f(a) \|\varphi\|^2, \quad (5.16)$$

$$\text{Re} \langle \varphi | e^F Y e^{-F} \varphi \rangle \geq (1 - g(a)) \langle \varphi | Y \varphi \rangle - h(a) \|\varphi\|^2. \quad (5.17)$$

Then we have, for all F satisfying (5.8),

$$\sup_{z \in J + i\mathbb{R}} \|e^F (Y - z)^{-1} e^{-F}\| \leq \delta^{-1}. \quad (5.18)$$

PROOF: We only sketch the proof and refer to Lemma 5.2 of [37] for the details. The assumptions $e^{\pm F} \mathcal{Q}(Y) \subset \mathcal{Q}(Y)$ and (5.16) ensure that the closure, Y_F , of $(e^F Y e^{-F}) \upharpoonright_{\mathcal{D}}$ agrees with the closed operator $e^F Y e^{-F}$. The bound (5.17) shows that the numerical range of Y_F is contained in the half space $\{z \in \mathbb{C} : \text{Re } z \geq \sup J + \delta\}$. Moreover, we can argue that, for $z \in J + i\mathbb{R}$, the deficiency of $Y_F - z$ is zero and, hence, the norm of $(Y_F - z)^{-1} = e^F (Y - z)^{-1} e^{-F}$ can be estimated by one over the distance of z to the numerical range of Y_F . \square

The semi-relativistic Pauli-Fierz operator. Next, we apply Lemma 5.5 to H_V^{PF} . In order to verify condition (5.17) with a good bound on the exponential decay rates we apply the following technical lemma from [25]:

Lemma 5.6 *For all $a \in (0, 1)$, F satisfying (5.8), and $\varphi \in \mathcal{D}$,*

$$\begin{aligned} \text{Re} \langle \varphi | e^F |D_{\mathbf{A}}| e^{-F} \varphi \rangle &\geq \langle \varphi | (D_{\mathbf{A}}^2 - |\nabla F|^2)^{1/2} \varphi \rangle \\ &\geq \langle \varphi | (|D_{\mathbf{A}}| - \rho(a)) \varphi \rangle. \end{aligned} \quad (5.19)$$

PROOF: For every $\varphi \in \mathcal{D}$, we infer from (4.12) that

$$\langle \varphi | (e^F |D_{\mathbf{A}}| e^{-F} - (D_{\mathbf{A}}^2 - |\nabla F|^2)^{1/2}) \varphi \rangle = \int_0^\infty J[\varphi; \eta] \frac{\eta^{1/2} d\eta}{\pi},$$

with

$$\begin{aligned} J[\varphi; \eta] &:= \langle \varphi | (\mathcal{R}_F(\eta) - e^F \mathcal{R}_0(\eta) e^{-F}) \varphi \rangle, \\ \mathcal{R}_G(\eta) &:= (D_{\mathbf{A}}^2 - |\nabla G|^2 + \eta)^{-1}, \quad G \in \{0, F\}. \end{aligned}$$

Now, let $\phi := e^F (D_{\mathbf{A}}^2 + \eta) e^{-F} \psi$, for some $\psi \in \mathcal{D}$. Then

$$\begin{aligned} \operatorname{Re} \langle \phi | e^F \mathcal{R}_0(\eta) e^{-F} \phi \rangle &= \operatorname{Re} \langle e^F (D_{\mathbf{A}}^2 + \eta) e^{-F} \psi | \psi \rangle \\ &= \langle (D_{\mathbf{A}}^2 - |\nabla F|^2 + \eta) \psi | \psi \rangle \\ &\geq (1 - a^2 + \eta) \|\psi\|^2 \geq 0. \end{aligned}$$

Since $D_{\mathbf{A}}^2$ is essentially self-adjoint on \mathcal{D} and multiplication with e^{-F} maps \mathcal{D} bijectively onto itself, we know that $(D_{\mathbf{A}}^2 + \eta) e^{-F} \mathcal{D}$ is dense in \mathcal{H} . Since F is bounded we conclude that the previous estimates hold, for all ϕ in some dense domain, whence $\operatorname{Re} [e^F \mathcal{R}_0(\eta) e^{-F}] \geq 0$ as a quadratic form on \mathcal{H} . Next, we set $Q := (\boldsymbol{\alpha} \cdot \nabla F) D_{\mathbf{A}} + D_{\mathbf{A}} (\boldsymbol{\alpha} \cdot \nabla F)$ and let

$$\varphi := (D_{\mathbf{A}}^2 - |\nabla F|^2 + \eta) \psi = e^{\pm F} (D_{\mathbf{A}}^2 + \eta) e^{\mp F} \psi \mp iQ\psi,$$

for $\psi \in \mathcal{D}$. Then

$$J[\varphi; \eta] = i \langle e^{-F} \mathcal{R}_0(\eta) e^F \varphi | Q \psi \rangle = i \langle \psi | Q \psi \rangle + \langle Q \psi | e^F \mathcal{R}_0(\eta) e^{-F} Q \psi \rangle.$$

Here $D_{\mathbf{A}}^2 - |\nabla F|^2$ is essentially self-adjoint on \mathcal{D} and Q is symmetric on the same domain. Hence, $\operatorname{Re} J[\varphi; \eta] \geq 0$, for all φ in a dense set, thus for all $\varphi \in \mathcal{H}$, and we arrive at the first inequality in (5.19). Since the square root is operator monotone, $|\nabla F| \leq a$, and $|D_{\mathbf{A}}| \geq 1$, we further have

$$(D_{\mathbf{A}}^2 - |\nabla F|^2)^{1/2} \geq |D_{\mathbf{A}}| + (D_{\mathbf{A}}^2 - a^2)^{1/2} - |D_{\mathbf{A}}| \geq |D_{\mathbf{A}}| - \rho(a).$$

□

In what follows we abbreviate

$$\mathcal{K}_F := [P_{\mathbf{A}}^+, e^F] e^{-F},$$

and recall from (2.24) that $\| |D_{\mathbf{A}}|^{1/2} \mathcal{K}_F \| = \mathcal{O}_{a_0}(a)$, for all F satisfying (5.8) with $0 < a \leq a_0 < 1$.

Lemma 5.7 *For all $0 < a \leq a_0 < 1$ and F satisfying (5.8),*

$$\operatorname{Re} \langle \varphi | e^F Y_V^{\text{PF}} e^{-F} \varphi \rangle \geq \langle \varphi | Y_V^{\text{PF}} \varphi \rangle - \rho(a) \|\varphi\|^2, \quad \varphi \in \mathcal{D}. \quad (5.20)$$

Moreover, for all $\varepsilon > 0$ and $a_0 \in (0, 1)$, there is some constant, $C(a_0, \varepsilon, V) \in (0, \infty)$, such that, for all F satisfying (5.8) with $a \in [0, a_0]$ and $\varphi \in \mathcal{D}$,

$$|\langle \varphi | (e^F Y_V^{\text{PF}} e^{-F} - Y_V^{\text{PF}}) \varphi \rangle| \leq \varepsilon \langle \varphi | Y_V^{\text{PF}} \varphi \rangle + C(a_0, \varepsilon, V) \|\varphi\|^2. \quad (5.21)$$

PROOF: (5.20) follows immediately from (5.19). To derive (5.21) we write

$$e^F |D_{\mathbf{A}}| e^{-F} - |D_{\mathbf{A}}| = -2 D_{\mathbf{A}} \mathcal{K}_F + i \boldsymbol{\alpha} \cdot (\nabla F) e^F S_{\mathbf{A}} e^{-F}$$

on \mathcal{D} and make a little observation. Since $F \equiv ar$ on $\mathcal{B}_{r/2}(0)$ we find some $\mu \in C^\infty(\mathbb{R}_x^3, [0, 1])$ such that $\mu = 0$ on $\overline{\mathcal{B}}_{r/4}(0)$ and $\nabla F = \mu \nabla F$. For $\varphi, \psi \in \mathcal{D}$, we thus have (recall (2.18) and (2.22))

$$|\langle D_{\mathbf{A}} \varphi | \mathcal{K}_F \psi \rangle| \leq \int_{\mathbb{R}} |\langle \varphi | D_{\mathbf{A}} R_{\mathbf{A}}(iy) \mu i \boldsymbol{\alpha} \cdot \nabla F R_{\mathbf{A}}^F(iy) \psi \rangle| \frac{dy}{2\pi}.$$

Here we can write

$$D_{\mathbf{A}} R_{\mathbf{A}}(iy) \mu = \mu |D_{\mathbf{A}}| S_{\mathbf{A}} R_{\mathbf{A}}(iy) + [D_{\mathbf{A}}, \mu] R_{\mathbf{A}}(iy) + D_{\mathbf{A}} R_{\mathbf{A}}(iy) [\mu, D_{\mathbf{A}}] R_{\mathbf{A}}(iy),$$

where $\|[D_{\mathbf{A}}, \mu]\| \leq \|\nabla \mu\|_\infty = \mathcal{O}(1)$. On account of

$$\| |D_{\mathbf{A}}|^{1/2} R_{\mathbf{A}}(iy) \| \leq \mathcal{O}(1) (1 + y^2)^{-1/4}$$

and $\|D_{\mathbf{A}} R_{\mathbf{A}}(iy)\| = \mathcal{O}(1)$ it is now straightforward to verify that

$$|\langle D_{\mathbf{A}} \varphi | \mathcal{K}_F \psi \rangle| \leq \mathcal{O}_{a_0}(a) \{ \| |D_{\mathbf{A}}|^{1/2} \mu \varphi \| + \|\varphi\| \} \|\psi\|.$$

Some elementary estimates using $\|\nabla F\|_\infty \leq a$, $\|e^F S_{\mathbf{A}} e^{-F}\| = \mathcal{O}_{a_0}(1)$, and the previous bound now show that

$$\begin{aligned} & |\langle \varphi | (e^F |D_{\mathbf{A}}| e^{-F} - |D_{\mathbf{A}}|) \varphi \rangle| \\ & \leq \varepsilon_1 \langle \mu \varphi | |D_{\mathbf{A}}| \mu \varphi \rangle + (\varepsilon_1^{-1} \mathcal{O}_{a_0}(a^2) + \mathcal{O}_{a_0}(a)) \|\varphi\|^2 \\ & \leq \varepsilon_1 \mathcal{O}(1) \langle \mu \varphi | Y_V^{\text{PF}} \mu \varphi \rangle + \text{const}(a_0, \varepsilon_1) \|\varphi\|^2, \end{aligned} \quad (5.22)$$

for every $\varepsilon_1 \in (0, 1]$. In the second step we used that μV is bounded because $\mu = 0$ on $\overline{\mathcal{B}}_{r/4}(0)$. Since we may assume that there is some $\tilde{\mu} \in C_0^\infty(\mathbb{R}^3, [0, 1])$ such that $\mu^2 + \tilde{\mu}^2 = 1$ we can employ an IMS localization formula as in the proof of Lemma 5.3 to show that $\mu Y_V^{\text{PF}} \mu \leq Y_V^{\text{PF}} + \mathcal{O}(1)$ on \mathcal{D} . Altogether this proves (5.21). \square

Lemma 5.8 *There exist constants, $c_1, c_2 \in (0, \infty)$, such that, for all $a \in (0, 1/2]$ and $\pm F$ satisfying (5.8), and $\varphi \in \mathcal{D}$,*

$$\langle e^F \varphi | Y_V^{\text{PF}} e^F \varphi \rangle \leq c_1 \|e^F\|^2 \langle \varphi | Y_V^{\text{PF}} \varphi \rangle + c_2 \|e^F\|^2 \|\varphi\|^2. \quad (5.23)$$

In particular, $e^F \mathcal{Q}(Y_V^{\text{PF}}) \subset \mathcal{Q}(Y_V^{\text{PF}})$.

PROOF: We pick a smooth partition of unity with respect to the electron coordinates, $\mu_0^2 + \mu_1^2 = 1$, where $\text{supp}(\mu_0) \subset \mathcal{B}_{r/2}(0)$ and $\mu_0 = 1$ on $\overline{\mathcal{B}}_{r/4}(0)$. Then $\langle e^F \varphi | Y_V^{\text{PF}} e^F \varphi \rangle = \sum_{i=0,1} \langle \mu_i e^F \varphi | Y_V^{\text{PF}} \mu_i e^F \varphi \rangle + R_\varphi$, where $|R_\varphi| \leq \mathcal{O}(1) \|e^F\|^2 \|\varphi\|^2$. (This holds in particular for $F = 0$, of course.) Therefore, it is sufficient to prove the bound (5.23) with $\varphi = \mu_i \psi$, $i = 0, 1$, $\psi \in \mathcal{D}$. For

$\varphi = \mu_0 \psi$, the bound holds, however, true trivially, for all $c_1, c_2 \geq 1$, since $F = 1$ on the support of μ_0 .

Let us assume that $\varphi = \mu_1 \psi$, for some $\psi \in \mathcal{D}$, in the rest of this proof. Of course, $\|\chi_{0,R} e^F \varphi\|^2 \leq \|e^F\|^2 \|\chi_{0,R} \varphi\|^2$ and, since H_f and e^F commute, $\|H_f^{1/2} e^F \varphi\|^2 \leq \|e^F\|^2 \|H_f^{1/2} \varphi\|^2$. Furthermore, $|\langle \varphi | V \varphi \rangle| \leq \mathcal{O}(1) \|\varphi\|^2$, since V is bounded on $\text{supp}(\mu_1)$. To conclude we write $|D_{\mathbf{A}}| = P_{\mathbf{A}}^+ D_{\mathbf{A}} - P_{\mathbf{A}}^- D_{\mathbf{A}}$ and employ the following bound derived in [39, Equation (4.24) and the succeeding paragraphs],

$$\langle \varphi | e^F P_{\mathbf{A}}^{\pm} (\pm D_{\mathbf{A}}) e^F \varphi \rangle \leq c_3 \|e^F\|^2 \langle \varphi | P_{\mathbf{A}}^{\pm} (\pm D_{\mathbf{A}}) \varphi \rangle + c_4 \|e^F\|^2 \|\varphi\|^2, \quad (5.24)$$

for every $\varphi \in \mathcal{D}$. We actually derived this bound in [39] for classical vector potentials. The proof works, however, also for the quantized vector potential without any change. Moreover, we only treated the choice of the plus sign in (5.24). But again an obvious modification of the proof in [39] shows that (5.24) is still valid when we choose the minus sign. \square

The no-pair operator. The following lemma implies that the conditions (5.16) and (5.17) are fulfilled in the case of the no-pair operator, too.

Lemma 5.9 *There exist $c \equiv c(V) \in (0, \infty)$ and $c' \equiv c'(V, d_{-1}, d_1, \Sigma^{\text{np}} - E_{V,R}^{\text{np}}) \in (0, \infty)$ such that, for all F satisfying (5.8),*

$$|\langle \varphi | (e^F Y_V^{\text{np}} e^{-F} - Y_V^{\text{np}}) \varphi \rangle| \leq \mathcal{O}(a) \langle \varphi | (c Y_V^{\text{np}} + c') \varphi \rangle, \quad \varphi \in \mathcal{D}.$$

PROOF: On account of (5.22) and $\|e^F \chi_{0,R} P_{\mathbf{A}}^+ \chi_{0,R} e^{-F} - \chi_{0,R} P_{\mathbf{A}}^+ \chi_{0,R}\| \leq \|\mathcal{K}_F\| = \mathcal{O}_{a_0}(a)$ it suffices to consider

$$\Delta^{\pm}(T) := e^F P_{\mathbf{A}}^{\pm} T P_{\mathbf{A}}^{\pm} e^{-F} - P_{\mathbf{A}}^{\pm} T P_{\mathbf{A}}^{\pm} = 2\text{Re} [P_{\mathbf{A}}^{\pm} T \delta P] + \delta P T \delta P,$$

where $\delta P := e^F P_{\mathbf{A}}^{\pm} e^{-F} - P_{\mathbf{A}}^{\pm}$ and T is H_f or V . Clearly,

$$|\langle \varphi | \Delta^{\pm}(T) \varphi \rangle| \leq \varepsilon \langle \varphi | P_{\mathbf{A}}^{\pm} T P_{\mathbf{A}}^{\pm} \varphi \rangle + (1 + \varepsilon^{-1}) \| |T|^{1/2} \delta P \varphi \|^2,$$

for all $\varepsilon > 0$ and $\varphi \in \mathcal{D}$. Since, by (2.25) and (2.26), $\| |T|^{1/2} \delta P \varphi \|^2 \leq \mathcal{O}(a^2) \langle \varphi | (H_f + (4d_1)^2 + 1) \varphi \rangle \leq \mathcal{O}(a^2) \langle \varphi | (H_f^{\text{diag}} + d_1^2 + 1) \varphi \rangle$, we may choose ε proportional to a and use (3.31) to conclude. \square

To complete the proof of Theorem 5.1 also in the case of the no-pair operator we note that the bound (5.23) still holds true when Y_V^{PF} is replaced by Y_0^{np} . To this end we only have to observe in addition to the remarks in the proof of Lemma 5.8 that $\|\check{H}_f^{1/2} P_{\mathbf{A}}^{\pm} e^F \varphi\| \leq \mathcal{O}(1) \|e^F\| \|\check{H}_f^{1/2} \varphi\|$. This follows, however, immediately from (9.9) which implies $\|\check{H}_f^{1/2} P_{\mathbf{A}}^{\pm} e^F \varphi\| \leq (1 + \|\mathcal{C}_{1/2}\|/2) \|e^F \check{H}_f^{1/2} \varphi\|$. Thus, $e^{\pm F} \mathcal{Q}(Y_0^{\text{np}}) \subset \mathcal{Q}(Y_0^{\text{np}})$. By Theorem 3.6 and the assumptions on V we know that $\mathcal{Q}(Y_V^{\text{np}}) = \mathcal{Q}(Y_0^{\text{np}})$ and we conclude.

6 Existence of ground states with mass

In this section we present an intermediate step of the proof of the existence of ground states for H_V^\sharp . Namely, we prove the existence of ground state eigenvectors, ϕ_m^\sharp , for modified Hamiltonians, $H_{V,m}^\sharp$, which are defined by means of an infra-red cut-off coupling function. The infra-red cut-off parameter, $m > 0$, is referred to as the photon mass. Later on in Section 8 we shall remove the infra-red cut-off by showing that every sequence, $\{\phi_{m_j}^\sharp\}_j$, $m_j \searrow 0$, contains a strongly convergent subsequence whose limit turns out to be a ground state eigenvector of H_V^\sharp . The compactness argument used to show this in Section 8 requires the infra-red bounds derived before in Section 7.

In the present section the existence of ϕ_m^\sharp is shown by discretizing the photon degrees of freedom. After the infra-red cut-off operators $H_{V,m}^\sharp$ have been defined in Subsection 6.1 we construct discretized versions of them, denoted by $H_{V,m,\varepsilon}^\sharp$, in Subsection 6.2. We collect some technical estimates needed to compare the original, infra-red cut-off, and discretized operators in Subsections 6.3 and 6.4. As another preparation we study the continuity of the ground state energy and ionization threshold with respect to the parameters m and ε in Subsection 6.5. The main result of this section, Theorem 6.11 on the existence of ϕ_m^\sharp , is stated and proved in Subsection 6.6 and we refer the reader to that subsection for some brief remarks on its proof. Many arguments of this section (in particular those in Subsections 6.4 and 6.5) are alternatives to the corresponding ones in [28, 29].

In the whole section $\mathbf{G}_\mathbf{x}$ is the coupling function given by (2.2). To clarify which properties of V are exploited we introduce the following hypothesis. It is fulfilled by the Coulomb potential in the subcritical cases:

Hypothesis 6.1 *In the case $\sharp = \text{PF}$ the potential $V \in L_{\text{loc}}^2(\mathbb{R}^3, \mathbb{R})$ is relatively form-bounded with respect to $\sqrt{-\Delta}$ with relative form bound strictly less than one. In the case $\sharp = \text{np}$ the potential $V \in L_{\text{loc}}^2(\mathbb{R}^3, \mathbb{R})$ satisfies $H^1(\mathbb{R}^3) \subset \mathcal{D}(V)$ and (3.30) with $a < 1$.*

We shall strengthen the assumptions on V later on in order to apply the localization estimates of Section 5.

6.1 Operators with photon mass

For every $m > 0$, the infra-red cut-off coupling function is given as

$$\mathbf{G}_{\mathbf{x},m}(k) := -e \frac{\mathbb{1}_{\{m \leq |\mathbf{k}| \leq \Lambda\}}}{2\pi\sqrt{|\mathbf{k}|}} e^{-i\mathbf{k} \cdot \mathbf{x}} \varepsilon(k), \quad (6.1)$$

for all $\mathbf{x} \in \mathbb{R}^3$ and almost every $k = (\mathbf{k}, \lambda) \in \mathbb{R}^3 \times \mathbb{Z}_2$. To compare $\mathbf{G}_{\mathbf{x},m}$ with $\mathbf{G}_\mathbf{x}$ defined in (2.2) we introduce the parameter

$$\Delta^2(m) := \int (\omega(k) + \omega(k)^{-1}) \sup_{\mathbf{x}} |\mathbf{G}_\mathbf{x}(k) - \mathbf{G}_{\mathbf{x},m}(k)|^2 dk. \quad (6.2)$$

Of course, $\Delta^2(m) = (e^2/2\pi^2) \int_{|\mathbf{k}| < m} (1 + |\mathbf{k}|^{-2}) d^3\mathbf{k} \rightarrow 0$, as $m \searrow 0$. For $m > 0$, we further define the infra-red cut-off vector potential,

$$\mathbf{A}_m := a^\dagger(\mathbf{G}_m) + a(\mathbf{G}_m), \quad a^\sharp(\mathbf{G}_m) := \int_{\mathbb{R}^3}^\oplus \mathbb{1}_{\mathbb{C}^4} \otimes a^\sharp(\mathbf{G}_{\mathbf{x},m}) d^3\mathbf{x},$$

and the infra-red regularized Hamiltonians

$$H_{V,m}^{\text{PF}} := |D_{\mathbf{A}_m}| + V + H_f, \quad (6.3)$$

$$H_{V,m}^{\text{np}} := P_{\mathbf{A}_m}^+ (D_{\mathbf{A}_m} + V + H_f) P_{\mathbf{A}_m}^+. \quad (6.4)$$

We define these operators as self-adjoint Friedrichs extensions starting from \mathcal{D} . The ground state energies and ionization thresholds, for positive photon mass $m > 0$, are denoted by

$$E_{V,m}^\sharp := \inf \sigma[H_{V,m}^\sharp], \quad \Sigma_m^\sharp := \inf \sigma[H_{0,m}^\sharp].$$

As a first step we introduce a truncated Fock space where the radiation field energy H_f is bounded from below by $m > 0$ on the orthogonal complement of the vacuum sector. Namely, we split the one-photon Hilbert space into two mutually orthogonal subspaces

$$\mathcal{H} = \mathcal{H}_m^> \oplus \mathcal{H}_m^<, \quad \mathcal{H}_m^> := L^2(\mathcal{A}_m \times \mathbb{Z}_2), \quad \mathcal{A}_m := \{|\mathbf{k}| \geq m\}.$$

It is well-known that $\mathcal{F}_b[\mathcal{H}] = \mathcal{F}_b[\mathcal{H}_m^>] \otimes \mathcal{F}_b[\mathcal{H}_m^<]$. We observe that \mathbf{A}_m creates and annihilates photon states in $\mathcal{H}_m^>$ only and H_f leaves the Fock space factors associated with the subspaces \mathcal{H}_m^{\leq} invariant. We shall designate operators acting in the Fock space factors $\mathcal{F}_b[\mathcal{H}_m^>]$ or $\mathcal{F}_b[\mathcal{H}_m^<]$ by a superscript $>$ or $<$, respectively. Under the isomorphism

$$\mathcal{H} \cong (L^2(\mathbb{R}^3, \mathbb{C}^4) \otimes \mathcal{F}_b[\mathcal{H}_m^>]) \otimes \mathcal{F}_b[\mathcal{H}_m^<] =: \mathcal{H}_m^> \otimes \mathcal{F}_b[\mathcal{H}_m^<], \quad (6.5)$$

we then have $D_{\mathbf{A}_m} \cong D_{\mathbf{A}_m^>} \otimes \mathbb{1}$, $|D_{\mathbf{A}_m}| \cong |D_{\mathbf{A}_m^>}| \otimes \mathbb{1}$, $P_{\mathbf{A}_m}^+ \cong P_{\mathbf{A}_m^>}^+ \otimes \mathbb{1}$, and $H_f = H_f^> \otimes \mathbb{1} + \mathbb{1} \otimes H_f^<$ with $H_{f,m}^> := d\Gamma(\omega|_{\mathcal{A}_m \times \mathbb{Z}_2})$, $H_{f,m}^< := d\Gamma(\omega|_{\mathcal{A}_m^c \times \mathbb{Z}_2})$. As a consequence, the semi-relativistic Pauli-Fierz and no-pair operators decompose under the isomorphism (6.5) as

$$\begin{aligned} H_{V,m}^{\text{PF}} &= \overline{H_{V,m,0}^{\text{PF}} \otimes \mathbb{1} + \mathbb{1} \otimes H_f^<}}, \\ H_{V,m}^{\text{np}} &= \overline{H_{V,m,0}^{\text{np}} \otimes \mathbb{1} + P_{\mathbf{A}_m^>}^+ \otimes H_f^<}}, \end{aligned} \quad (6.6)$$

where

$$\begin{aligned} H_{V,m,0}^{\text{PF}} &:= |D_{\mathbf{A}_m^>}| + V + H_f^>, \\ H_{V,m,0}^{\text{np}} &:= P_{\mathbf{A}_m^>}^+ (D_{\mathbf{A}_m^>} + V + H_f^>) P_{\mathbf{A}_m^>}^+. \end{aligned}$$

The latter operators act in the Hilbert spaces $\mathcal{H}_m^>$ and $P_{\mathbf{A}_m^>}^+ \mathcal{H}_m^>$, respectively. The following lemma [28, 29] shows in particular that it suffices to prove the existence of ground states in these truncated Hilbert spaces.

Lemma 6.2 *Assume that V fulfills Hypothesis 6.1. Then, for all $m > 0$,*

$$E_{V,m}^\sharp = \inf \sigma[H_{V,m,0}^\sharp], \quad \Sigma_m^\sharp = \inf \sigma[H_{0,m,0}^\sharp].$$

Moreover, if $E_{V,m}^\sharp$ is an eigenvalue of $H_{V,m,0}^\sharp$, then it is an eigenvalue of $H_{V,m}^\sharp$, too.

PROOF: It is clear that $H_{V,m,0}^\sharp \otimes \mathbb{1} \leq H_{V,m}^\sharp$, whence $\inf \sigma[H_{V,m,0}^\sharp] \leq \inf \sigma[H_{V,m}^\sharp]$. Next, we pick a minimizing sequence of normalized vectors $\psi_n^\triangleright \in \mathcal{Q}(H_{V,m,0}^\sharp)$, $\langle \psi_n^\triangleright | H_{V,m,0}^\sharp \psi_n^\triangleright \rangle \rightarrow \inf \sigma[H_{V,m,0}^\sharp]$. Setting $\psi_n := \psi_n^\triangleright \otimes \Omega^\triangleright$, where Ω^\triangleright is the vacuum vector in $\mathcal{F}_b[\mathcal{H}_m^\triangleright]$, we observe that

$$\langle \psi_n | H_{V,m}^\sharp \psi_n \rangle = \langle \psi_n^\triangleright | H_{V,m,0}^\sharp \psi_n^\triangleright \rangle,$$

thus $\inf \sigma[H_{V,m,0}^\sharp] \geq \inf \sigma[H_{V,m}^\sharp]$. Likewise, if $\phi_m^\sharp \in \mathcal{H}_m^\triangleright$ is a ground state eigenvector of $H_{V,m,0}^\sharp$, then $\phi_m^\sharp \otimes \Omega^\triangleright$ is a ground state eigenvector of $H_{V,m}^\sharp$. \square

In order to show the existence of a ground state for $H_{V,m,0}^\sharp$ in the next step it is sufficient to show that the spectrum of $H_{V,m,0}^\sharp$ is discrete in a neighborhood of $E_{V,m}^\sharp$. ($E_{V,m}^\sharp$ is contained in the essential spectrum of $H_{V,m}^\sharp$ on the contrary.) A general strategy to achieve this would be the following. We could seek for a self-adjoint operator, A , satisfying $-\infty < A \leq H_{V,m,0}^\sharp$ and having discrete spectrum in $(-\infty, E_{V,m}^\sharp + c]$, for some $c > 0$. If such an operator A exists then also $H_{V,m,0}^\sharp$ has discrete spectrum in $(-\infty, E_{V,m}^\sharp + c]$. We need, however, a modification of this strategy. Let χ denote the spectral projection of $H_{V,m,0}^\sharp$ corresponding to some half-line $(-\infty, E_{V,m}^\sharp + c]$, $c > 0$. Then we seek for a self-adjoint auxiliary operator A such that

$$\chi A \chi \leq \chi \{H_{V,m,0}^\sharp - E_{V,m}^\sharp - 2c\} \chi \leq -c \chi \quad (6.7)$$

and $\text{Tr}\{\chi A \chi\} > -\infty$, where Tr denotes the trace. For in this case we have $\text{Tr}\{\chi\} < \infty$. The latter strategy is advantageous since we only have to compare A and $H_{V,m,0}^\sharp$ on the range of χ whose elements are exponentially localized by Theorem 5.1, provided that $c > 0$ is appropriately chosen. A suitable comparison operator A is constructed by means of a discretization of the photon momenta in the next subsection.

6.2 Discretization of the photon momenta

On $\mathcal{H}_m^\triangleright$ we introduce a discretization in the photon momenta: For every $\varepsilon > 0$, we decompose $\mathcal{A}_m = \{|\mathbf{k}| \geq m\}$ as

$$\mathcal{A}_m = \bigcup_{\nu \in (\varepsilon\mathbb{Z})^3} Q_m^\varepsilon(\nu), \quad Q_m^\varepsilon(\nu) := (\nu + [-\varepsilon/2, \varepsilon/2)^3) \cap \mathcal{A}_m, \quad \nu \in (\varepsilon\mathbb{Z})^3.$$

Of course, for every $\mathbf{k} \in \mathcal{A}_m$, we find a unique vector, $\tilde{\nu}_\varepsilon(\mathbf{k}) \in (\varepsilon\mathbb{Z})^3$, such that $\mathbf{k} \in Q_m^\varepsilon(\tilde{\nu}_\varepsilon(\mathbf{k}))$. To each $\nu \in (\varepsilon\mathbb{Z})^3$ with $Q_m^\varepsilon(\nu) \neq \emptyset$ we further associate some $\varkappa_{m,\varepsilon}(\nu) \in Q_m^\varepsilon(\nu)$ such that

$$|\varkappa_{m,\varepsilon}(\nu)| = \inf_{\mathbf{k} \in Q_m^\varepsilon(\nu)} |\mathbf{k}|.$$

In this way we obtain a map

$$\nu_\varepsilon : \mathcal{A}_m \times \mathbb{Z}_2 \longrightarrow \mathbb{R}^3, \quad k = (\mathbf{k}, \lambda) \longmapsto \nu_\varepsilon(k) := \varkappa_{m,\varepsilon}(\tilde{\nu}_\varepsilon(\mathbf{k})). \quad (6.8)$$

It is evident that the vectors $\varkappa_{m,\varepsilon}(\nu)$ can be chosen such that

$$\nu_\varepsilon(-\mathbf{k}, \lambda) = -\nu_\varepsilon(\mathbf{k}, \lambda), \quad \text{for almost every } \mathbf{k} \in \mathcal{A}_m. \quad (6.9)$$

The set of Lebesgue measure zero where the identity (6.9) might not hold is contained in the union of all planes which are perpendicular to some coordinate axis and contain points of the lattice $(\varepsilon\mathbb{Z})^3$. We define the ε -average of a locally integrable function, f , on $\mathcal{A}_m \times \mathbb{Z}_2$ by

$$[P_\varepsilon f](k) := \frac{1}{|Q_m^\varepsilon(\tilde{\nu}_\varepsilon(\mathbf{k}))|} \int_{Q_m^\varepsilon(\tilde{\nu}_\varepsilon(\mathbf{k}))} f(\mathbf{p}, \lambda) d^3\mathbf{p}, \quad (6.10)$$

and introduce the following discretized coupling function,

$$\mathbf{G}_{\mathbf{x},m,\varepsilon}(k) := -(e/2\pi) e^{-i\nu_\varepsilon(k) \cdot \mathbf{x}} P_\varepsilon [\mathbb{1}_{\mathcal{A}_m} \omega^{-1/2} \varepsilon](k),$$

for all $\mathbf{x} \in \mathbb{R}^3$ and almost every $k = (\mathbf{k}, \lambda) \in \mathcal{A}_m \times \mathbb{Z}_2$. In order to compare $\mathbf{G}_{\mathbf{x},m}$ with $\mathbf{G}_{\mathbf{x},m,\varepsilon}$ we put

$$\Delta_*^2(a, m, \varepsilon) := \int_{\mathcal{A}_m \times \mathbb{Z}_2} (\omega(k) + \omega^{-1}(k)) \sup_{\mathbf{x}} \{e^{-a|\mathbf{x}|} |\mathbf{G}_{\mathbf{x},m}(k) - \mathbf{G}_{\mathbf{x},m,\varepsilon}(k)|^2\} dk, \quad (6.11)$$

for $a, m, \varepsilon > 0$. It is elementary to verify that $\Delta_*(a, m, \varepsilon) \rightarrow 0$, as $\varepsilon \searrow 0$, for fixed $a, m > 0$. Notice that $\Delta_*(a, m, \varepsilon)$ did not converge to zero if we chose $a = 0$. The fact that we need some weight function in \mathbf{x} to control the difference between $\mathbf{G}_{\mathbf{x},m}$ and $\mathbf{G}_{\mathbf{x},m,\varepsilon}$ is one of the reasons why a localization estimate for spectral subspaces is required to prove the existence of ground states. The discretized vector potential is now given as

$$\mathbf{A}_{m,\varepsilon} := a^\dagger(\mathbf{G}_{m,\varepsilon}) + a(\mathbf{G}_{m,\varepsilon}), \quad a^\sharp(\mathbf{G}_{m,\varepsilon}) := \int_{\mathbb{R}^3}^\oplus \mathbb{1}_{\mathbb{C}^4} \otimes a^\sharp(\mathbf{G}_{\mathbf{x},m,\varepsilon}) d^3\mathbf{x}.$$

The reason why we choose vectors ν_ε fulfilling (6.9) is its consequence

$$\mathbf{G}_{\mathbf{x},m,\varepsilon}(-\mathbf{k}, \lambda) = \overline{\mathbf{G}_{\mathbf{x},m,\varepsilon}(\mathbf{k}, \lambda)}.$$

The latter identity ensures that different components, $\mathbf{A}_{m,\varepsilon}^{(i)}(\mathbf{x}), \mathbf{A}_{m,\varepsilon}^{(j)}(\mathbf{y})$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, $i, j \in \{1, 2, 3\}$, of the discretized vector potential still commute. We have used this property in Section 3. The dispersion relation is discretized as

$$\omega_\varepsilon(k) := \inf \{ |\mathbf{p}| : \mathbf{p} \in Q_m^\varepsilon(\tilde{\nu}_\varepsilon(\mathbf{k})) \}, \quad k = (\mathbf{k}, \lambda) \in \mathcal{A}_m \times \mathbb{Z}_2.$$

Then $|\nu_\varepsilon| \leq \omega_\varepsilon$ on $\mathcal{A}_m \times \mathbb{Z}_2$ and

$$m \leq \omega_\varepsilon \leq \omega \text{ on } \mathcal{A}_m \times \mathbb{Z}_2, \quad H_{f,m,\varepsilon} := d\Gamma(\omega_\varepsilon) \leq H_{f,m}^>. \quad (6.12)$$

Here the operators $H_{f,m,\varepsilon}$ and $H_{f,m}^>$ are acting in $\mathcal{F}_b[\mathcal{K}_m^>]$. Finally, we define discretized Hamiltonians, $H_{V,m,\varepsilon}^\sharp$, acting in $L^2(\mathbb{R}^2, \mathbb{C}^4) \otimes \mathcal{F}_b[\mathcal{K}_m^>]$,

$$\begin{aligned} H_{V,m,\varepsilon}^{\text{PF}} &:= |D_{\mathbf{A}_{m,\varepsilon}}| + V + H_{f,m,\varepsilon}, \\ H_{V,m,\varepsilon}^{\text{np}} &:= P_{\mathbf{A}_{m,\varepsilon}}^+ (D_{\mathbf{A}_{m,\varepsilon}} + V + H_{f,m,\varepsilon}) P_{\mathbf{A}_{m,\varepsilon}}^+. \end{aligned}$$

6.3 Comparison of operators with different coupling functions

In order to compare the various modified operators we derive some bounds on differences of projections whose proofs are essentially consequences of the ideas collected in Subsection 2.4 and the bounds

$$\|\alpha \cdot (\mathbf{A} - \mathbf{A}_m) \check{H}_f^{-1/2}\| = \mathcal{O}(\Delta(m)), \quad (6.13)$$

$$\|\alpha \cdot (\mathbf{A}_m^> - \mathbf{A}_{m,\varepsilon}) \check{H}_f^{-1/2} e^{-a|\mathbf{x}|}\| = \mathcal{O}(\Delta_*(a, m, \varepsilon)). \quad (6.14)$$

Here we use the notation (6.2) and (6.11).

Lemma 6.3 *Let V be a symmetric multiplication operator in $L^2(\mathbb{R}^3)$ which is relatively form bounded with respect to $\sqrt{-\Delta}$.*

(i) *Set $\check{H}_f = H_f + E$, for some sufficiently large $E \geq 1$ depending on e and Λ , and let $\nu \geq 0$. Then, as $m \searrow 0$,*

$$\| |D_{\mathbf{A}}|^{1/2} (P_{\mathbf{A}}^\pm - P_{\mathbf{A}_m}^\pm) \check{H}_f^{-1/2} \| \leq \mathcal{O}(\Delta(m)), \quad (6.15)$$

$$\| |D_{\mathbf{A}_m}|^{1/2} (P_{\mathbf{A}}^\pm - P_{\mathbf{A}_m}^\pm) \check{H}_f^{-1/2} \| \leq \mathcal{O}(\Delta(m)), \quad (6.16)$$

$$\| \check{H}_f^\nu (P_{\mathbf{A}}^\pm - P_{\mathbf{A}_m}^\pm) \check{H}_f^{-\nu-1/2} \| \leq \mathcal{O}(\Delta(m)), \quad (6.17)$$

$$\| |V|^{1/2} (P_{\mathbf{A}}^\pm - P_{\mathbf{A}_m}^\pm) \check{H}_f^{-1} \| \leq \mathcal{O}(\Delta(m)). \quad (6.18)$$

(ii) *Let \check{H}_f be $H_{f,m}^> + E$ or $H_{f,m,\varepsilon} + E$, for some sufficiently large $E \geq 1$ depending on e and Λ , and let $\nu \geq 0$ and $a_0 \in (0, 1)$. Then, for every $a \in (0, a_0]$ and $F \in C^\infty(\mathbb{R}_{\mathbf{x}}^3, [0, \infty))$ satisfying $|\nabla F| \leq a$, $F(\mathbf{x}) \geq a|\mathbf{x}|$, for all $\mathbf{x} \in \mathbb{R}^3$, and $F(\mathbf{x}) = a|\mathbf{x}|$, for large $|\mathbf{x}|$, and for all sufficiently small $m, \varepsilon > 0$,*

$$\| |D_{\mathbf{A}_m^>}|^{1/2} (P_{\mathbf{A}_m^>}^\pm - P_{\mathbf{A}_{m,\varepsilon}}^\pm) \check{H}_f^{-1/2} e^{-F} \| \leq \mathcal{O}(\Delta_*(a, m, \varepsilon)), \quad (6.19)$$

$$\| |D_{\mathbf{A}_{m,\varepsilon}}|^{1/2} (P_{\mathbf{A}_m^>}^\pm - P_{\mathbf{A}_{m,\varepsilon}}^\pm) \check{H}_f^{-1/2} e^{-F} \| \leq \mathcal{O}(\Delta_*(a, m, \varepsilon)), \quad (6.20)$$

$$\| \check{H}_f^\nu (P_{\mathbf{A}_m^>}^\pm - P_{\mathbf{A}_{m,\varepsilon}}^\pm) \check{H}_f^{-\nu-1/2} e^{-F} \| \leq \mathcal{O}(\Delta_*(a, m, \varepsilon)), \quad (6.21)$$

$$\| |V|^{1/2} (P_{\mathbf{A}_m^>}^\pm - P_{\mathbf{A}_{m,\varepsilon}}^\pm) \check{H}_f^{-1} e^{-F} \| \leq \mathcal{O}(\Delta_*(a, m, \varepsilon)). \quad (6.22)$$

PROOF: By the assumption on V and Theorem 3.4 we have $|V| \leq C(|D_{\mathbf{A}}| + \check{H}_{\mathbf{f}})$. Therefore, the bounds (6.18) and (6.22) are consequences of (6.15)&(6.17) and (6.19)&(6.21), respectively. In order to prove the remaining estimates we pick two vector potentials, \mathbf{A}_1 and \mathbf{A}_2 , such that the set $\{\mathbf{A}_1, \mathbf{A}_2\}$ equals either $\{\mathbf{A}, \mathbf{A}_m\}$ (in which case $F := 0$ in what follows) or $\{\mathbf{A}_m^>, \mathbf{A}_{m,\varepsilon}\}$. For $j = 1, 2$ and $y \in \mathbb{R}$, we set $R_{\mathbf{A}_j}(iy) := (D_{\mathbf{A}_j} - iy)^{-1}$ and $R_{\mathbf{A}_j}^F(iy) := (D_{\mathbf{A}_j} + i\boldsymbol{\alpha} \cdot \nabla F - iy)^{-1}$. (Recall (2.20) and (2.21).) Then we have the following resolvent identity, for $\mu \geq 1/2$ and $y \in \mathbb{R}$,

$$\begin{aligned} & (R_{\mathbf{A}_1}(iy) - R_{\mathbf{A}_2}(iy)) e^{-F} \check{H}_{\mathbf{f}}^{-\mu} \\ &= R_{\mathbf{A}_1}(iy) \boldsymbol{\alpha} \cdot (\mathbf{A}_2 - \mathbf{A}_1) \check{H}_{\mathbf{f}}^{-\mu} e^{-F} R_{\mathbf{A}_2}^F(iy) \Upsilon_{\mu}^F(iy), \end{aligned}$$

where $\Upsilon_{\mu}^F(iy)$ is the bounded operator defined in (9.5) below (with \mathbf{A} replaced by \mathbf{A}_1). Using (2.18) we find, for all ϕ and f in the Hilbert space,

$$\begin{aligned} & |\langle f | (P_{\mathbf{A}_1}^{\pm} - P_{\mathbf{A}_2}^{\pm}) \check{H}_{\mathbf{f}}^{-\mu} e^{-F} \phi \rangle| \\ & \leq \int_{\mathbb{R}} |\langle f | R_{\mathbf{A}_1}(iy) \boldsymbol{\alpha} \cdot (\mathbf{A}_2 - \mathbf{A}_1) \check{H}_{\mathbf{f}}^{-\mu} e^{-F} R_{\mathbf{A}_2}^F(iy) \Upsilon_{\mu}^F(iy) \phi \rangle| \frac{dy}{\pi}. \end{aligned} \quad (6.23)$$

In the case $\mu = 1/2$, we choose $f = |D_{\mathbf{A}_1}|^{1/2} \psi$, $\psi \in \mathcal{D}(|D_{\mathbf{A}_1}|^{1/2})$, and observe that the integrand in (6.23) is then bounded by

$$\mathcal{O}_* \| |D_{\mathbf{A}_1}|^{1/2} R_{\mathbf{A}_1}(iy) \| \| R_{\mathbf{A}_2, L}(iy) \| \| \Upsilon_{1/2, L}(iy) \| \| \phi \| \| \psi \|,$$

which is integrable due to (2.21), (6.13), (6.14), and the spectral theorem. Here and below $\mathcal{O}_* = \mathcal{O}(\Delta(m))$ or $\mathcal{O}_* = \mathcal{O}(\Delta_*(a, m, \varepsilon))$ depending on the choice of \mathbf{A}_j . This concludes the proof of (6.15), (6.16), (6.19), and (6.20).

In order to prove (6.17) and (6.21) we infer from Lemma 9.1 that the commutator $T_{\nu} := \check{H}_{\mathbf{f}}^{\nu} [\boldsymbol{\alpha} \cdot (\mathbf{A}_2 - \mathbf{A}_1) e^{-F}, \check{H}_{\mathbf{f}}^{-\nu}]$ extends to a bounded operator with $\|T_{\nu}\| = \mathcal{O}_*$. Together with (6.13), (6.14), and (9.7) this implies, for $\nu \geq 0$ and $\psi, \phi \in \mathcal{D}(H_{\mathbf{f}}^{\nu})$,

$$\begin{aligned} & |\langle \check{H}_{\mathbf{f}}^{\nu} \psi | R_{\mathbf{A}_1}(iy) \boldsymbol{\alpha} \cdot (\mathbf{A}_2 - \mathbf{A}_1) \check{H}_{\mathbf{f}}^{-\nu-1/2} e^{-F} R_{\mathbf{A}_2}^F(iy) \Upsilon_{\nu+1/2}^F(iy) \phi \rangle| \\ &= |\langle \psi | R_{\mathbf{A}_1}(iy) \Upsilon_{\nu}^F(iy) \check{H}_{\mathbf{f}}^{\nu} \boldsymbol{\alpha} \cdot (\mathbf{A}_2 - \mathbf{A}_1) \check{H}_{\mathbf{f}}^{-\nu-1/2} e^{-F} \times \\ & \quad \times R_{\mathbf{A}_2}^F(iy) \Upsilon_{\nu+1/2}^F(iy) \phi \rangle| \\ &\leq \mathcal{O}_* \| R_{\mathbf{A}_1}(iy) \| \| \Upsilon_{\nu}^F(iy) \| \| R_{\mathbf{A}_2}^F(iy) \| \| \Upsilon_{\nu+1/2}^F(iy) \| \| \psi \| \| \phi \| \\ &\leq \mathcal{O}_* (1 + y^2)^{-1} \| \psi \| \| \phi \|. \end{aligned}$$

Therefore, (6.17) and (6.21) follow from (6.23) upon choosing $\mu = \nu + 1/2$ and $f = \check{H}_{\mathbf{f}}^{\nu} \psi$. \square

6.4 Higher order estimates and their consequences

As a preparation for the proof of the existence of ground states for $H_{V, m, 0}^{\sharp}$ we derive bounds on certain expectation values of the difference $H_{V, m, 0}^{\sharp} - H_{V, m, \varepsilon}^{\sharp}$ in

this subsection. Moreover, we compare the ground state energies and ionization thresholds of $H_{V,m,0}^\sharp$ with those of $H_{V,m,\varepsilon}^\sharp$ defined by

$$E_{V,m,\varepsilon}^\sharp := \inf \sigma[H_{V,m,\varepsilon}^\sharp], \quad \Sigma_{m,\varepsilon}^\sharp := \inf \sigma[H_{0,m,\varepsilon}^\sharp],$$

for $m, \varepsilon > 0$. It is not possible to compare the no-pair operators $H_{V,m,0}^{\text{np}}$ and $H_{V,m,\varepsilon}^{\text{np}}$ in a quadratic form sense. For some error terms in the difference of these two operators “have the size of $|\mathbf{x}| H_f^{3/2}$ ” and can only be controlled when we take expectations with respect to states in some low-lying exponentially localized spectral subspace. To control the higher power $H_f^{3/2}$ of the radiation field energy we need, however, yet another non-trivial ingredient, namely the higher order estimates of the next theorem. Because of lack of space we cannot comment on the proof of Theorem 6.4 and refer to [36] instead. We remark that, for the semi-relativistic Pauli-Fierz operator, higher order estimates have been obtained earlier in [14]. Their proof given in [36] is, however, different and more model-independent so that the no-pair operator can also be treated along the same lines in [36]. In the case of the no-pair operator only the Coulomb potential is considered in [36]. An inspection of the proofs in [36] shows, however, that they immediately extend to all potentials satisfying Hypothesis 6.1.

Theorem 6.4 *Let $e \in \mathbb{R}$, $\Lambda \in (0, \infty)$, and assume that V fulfills Hypothesis 6.1. Then $\mathcal{D}((H_{V,m,\varepsilon}^\sharp)^{n/2}) \subset \mathcal{D}(H_f^{n/2})$, for every $n \in \mathbb{N}$, and there exist constants, $\varepsilon_0, m_0, C \in (0, \infty)$, such that, for all $\varepsilon \in [0, \varepsilon_0]$ and $m \in (0, m_0]$,*

$$\begin{aligned} \|H_{f,m,\varepsilon}^{n/2} \upharpoonright_{P_{\mathbf{A},m,\varepsilon}^+ \mathcal{H}_m^>} (H_{V,m,\varepsilon}^{\text{np}} - (E_{V,m,\varepsilon}^{\text{np}} - 1)P_{\mathbf{A},m,\varepsilon}^+)^{-n/2}\| &\leq C(1 + |E_{V,m,\varepsilon}^{\text{np}}|)^{2n}, \\ \|H_{f,m,\varepsilon}^{n/2} (H_{V,m,\varepsilon}^{\text{PF}} - (E_{V,m,\varepsilon}^{\text{PF}} - 1))^{-n/2}\| &\leq C(1 + |E_{V,m,\varepsilon}^{\text{PF}}|)^{2n}. \end{aligned}$$

If $V = 0$, then $E_{V,m,\varepsilon}^\sharp$ has to be replaced by $\Sigma_{m,\varepsilon}^\sharp$ in these bounds. Analogous bounds hold for H_V^\sharp .

Lemma 6.5 *Let $e \in \mathbb{R}$, $\Lambda \in (0, \infty)$, and assume that V fulfills Hypothesis 6.1. Then we find some $m_0 > 0$ such that the following holds:*

(i) *For all $m \in (0, m_0]$ and $\psi^+ \in \text{Ran}(\mathbb{1}_{(-\infty, \Sigma_m^\sharp+1]}(H_V^\sharp))$,*

$$|\langle \psi^+ | H_{V,m}^\sharp \psi^+ \rangle - \langle \psi^+ | H_V^\sharp \psi^+ \rangle| \leq \text{const}(\Sigma_m^\sharp, |E_V^\sharp|) o(m^0) \|\psi^+\|^2.$$

(ii) *For all $m \in (0, m_0]$ and $\psi^+ \in \text{Ran}(\mathbb{1}_{(-\infty, \Sigma_m^\sharp+1]}(H_{V,m}^\sharp))$,*

$$|\langle \psi^+ | H_{V,m}^\sharp \psi^+ \rangle - \langle \psi^+ | H_V^\sharp \psi^+ \rangle| \leq \text{const}(\Sigma_m^\sharp, |E_{V,m}^\sharp|) o(m^0) \|\psi^+\|^2.$$

PROOF: We treat only the no-pair operator explicitly. On account of the formula $|D_{\mathbf{A}}| = P_{\mathbf{A}}^+ D_{\mathbf{A}} - P_{\mathbf{A}}^- D_{\mathbf{A}}$ it will then be clear how to obtain the result also for the semi-relativistic Pauli-Fierz operator. We remark only once that, for instance, the inclusion $P_{\mathbf{A},m}^+ \text{Ran}(\mathbb{1}_{(-\infty, \Sigma_m^\sharp+1]}(H_V^\sharp)) \subset \mathcal{Q}(H_{V,m}^\sharp)$ follows from

the characterization of the form domains in Theorems 3.4 and 3.6. In the rest of this section we shall use similar remarks without further notice to simplify the exposition. Let $\delta P := P_{\mathbf{A}}^+ - P_{\mathbf{A}_m}^+$ and $\mathcal{M} := \{D_{\mathbf{A}}, V, H_f\}$. Then we have

$$H_V^{\text{np}} - H_{V,m}^{\text{np}} = P_{\mathbf{A}_m}^+ \alpha \cdot (\mathbf{A} - \mathbf{A}_m) P_{\mathbf{A}_m}^+ + \sum_{T \in \mathcal{M}} \{2\text{Re} [P_{\mathbf{A}}^+ T \delta P] - \delta P T \delta P\} \quad (6.24)$$

in the sense of quadratic forms on $\mathcal{Q}(|D_{\mathbf{0}}|) \cap \mathcal{Q}(H_f)$. Now, let

$$\psi^+ \in \text{Ran}(\mathbb{1}_{(-\infty, \Sigma^\sharp+1]}(H_V^\sharp)).$$

From (3.16), (6.13), (6.15)–(6.18), and (6.24) we readily infer that

$$|\langle \psi^+ | (H_V^{\text{np}} - H_{V,m}^{\text{np}}) \psi^+ \rangle| \leq \mathcal{O}(\triangle(m)) \{ \| |D_{\mathbf{A}}|^{1/2} \psi^+ \|^2 + \| \check{H}_f \psi^+ \|^2 \}.$$

Here $\| |D_{\mathbf{A}}|^{1/2} \psi^+ \| \leq \mathcal{O}(1) \|\psi^+\|$ since ψ^+ belongs to the spectral subspace $\text{Ran}(\mathbb{1}_{(-\infty, \Sigma^\sharp+1]}(H_V^\sharp))$ and the term containing V is a small form perturbation of H_V^\sharp by Hypothesis 6.1 and Theorem 3.4 or Theorem 3.6, respectively. Moreover, $\| \check{H}_f \psi^+ \| \leq \text{const}(\Sigma, |E_V|) \|\psi^+\|$ because of the higher order estimates. We can argue analogously if ψ^+ belongs to a spectral subspace of $H_{V,m}^\sharp$. In this case the right hand side of the estimate depends on Σ_m and $|E_{V,m}|$ since we apply higher order estimates for $H_{V,m}^\sharp$. \square

We recall once more that some \mathbf{x} -dependent weight is required to control the difference between $\mathbf{A}_m^>$ and $\mathbf{A}_{m,\varepsilon}$. If we consider only vectors ψ^+ in a spectral subspace corresponding to sufficiently low energies of $H_{V,m,\varepsilon}^\sharp$, then we can borrow this weight from the exponential localization of ψ^+ . At this point we have to introduce further conditions on the potential in order to guarantee that *there are* non-trivial spectral subspaces below the ionization threshold.

Hypothesis 6.6 *V satisfies (5.3) and there exist $c, m_\star, \varepsilon_\star > 0$ such that, for all $m \in (0, m_\star]$ and all $\varepsilon \in (0, \varepsilon_\star]$,*

$$\Sigma^\sharp - E_V^\sharp \geq c, \quad \Sigma_m^\sharp - E_{V,m}^\sharp \geq c, \quad \Sigma_{m,\varepsilon}^\sharp - E_{V,m,\varepsilon}^\sharp \geq c. \quad (6.25)$$

Examples of potentials fulfilling the previous hypothesis have been found in Theorem 4.1. In fact, as already pointed out there, the proof of Theorem 4.1 works also for the discretized operators when the unitary transformation employed in Section 4 is modified suitably; see [28, 29] for details. There are, however, potentials fulfilling Hypotheses 6.1 and 6.6 which are not covered by Theorem 4.1, for instance, those mentioned in Remark 4.2. This is the reason why we work with the very implicit Hypothesis 6.6 in what follows.

Lemma 6.7 *Let $e \in \mathbb{R}$, $\Lambda \in (0, \infty)$, and assume that V fulfills Hypotheses 6.1 and 6.6. Then there exist $\varepsilon_0, m_0 > 0$ such that the following holds, for all $\varepsilon \in (0, \varepsilon_0]$ and $m \in (0, m_0]$:*

(i) For all $\lambda \in (E_{V,m}^\sharp, \Sigma_m^\sharp)$ and $\psi^+ \in \text{Ran}(\mathbb{1}_{(-\infty, \lambda]}(H_{V,m}^\sharp))$,

$$\begin{aligned} & |\langle \psi^+ | H_{V,m,\varepsilon}^\sharp \psi^+ \rangle - \langle \psi^+ | H_{V,m,0}^\sharp \psi^+ \rangle| \\ & \leq \text{const}(\Sigma_m^\sharp, |E_{V,m}^\sharp|, (\Sigma_m^\sharp - \lambda)^{-1}) o(\varepsilon^0) \|\psi^+\|^2. \end{aligned} \quad (6.26)$$

(ii) For all $\lambda \in (E_{V,m,\varepsilon}^\sharp, \Sigma_{m,\varepsilon}^\sharp)$ and $\psi^+ \in \text{Ran}(\mathbb{1}_{(-\infty, \lambda]}(H_{V,m,\varepsilon}^\sharp))$,

$$\begin{aligned} & |\langle \psi^+ | H_{V,m,\varepsilon}^\sharp \psi^+ \rangle - \langle \psi^+ | H_{V,m,0}^\sharp \psi^+ \rangle| \\ & \leq \text{const}(\Sigma_{m,\varepsilon}^\sharp, |E_{V,m,\varepsilon}^\sharp|, (\Sigma_{m,\varepsilon}^\sharp - \lambda)^{-1}) o(\varepsilon^0) \|\psi^+\|^2. \end{aligned} \quad (6.27)$$

PROOF: Again we only treat the no-pair operator since the proofs for the semi-relativistic Pauli-Fierz operator will then be obvious.

(i): We have a formula for $H_{V,m,0}^{\text{np}} - H_{V,m,\varepsilon}^{\text{np}}$ similar to (6.24) with $(\mathbf{A}, \mathbf{A}_m, H_f)$ replaced by $(\mathbf{A}_m^>, \mathbf{A}_{m,\varepsilon}, H_{f,m}^>)$ where one additional term has to be added, namely $P_{\mathbf{A}_{m,\varepsilon}}^+ d\Gamma(\omega - \omega_\varepsilon) P_{\mathbf{A}_{m,\varepsilon}}^+$. Using this formula, (6.14), (6.19)–(6.22), and $|\omega - \omega_\varepsilon| \leq \sqrt{3}\varepsilon \leq (\sqrt{3}\varepsilon/m)\omega_\varepsilon \leq (\sqrt{3}\varepsilon/m)\omega$, for $|\mathbf{k}| \geq m$, which yields

$$|\langle \psi^+ | P_{\mathbf{A}_{m,\varepsilon}}^+ d\Gamma(\omega - \omega_\varepsilon) P_{\mathbf{A}_{m,\varepsilon}}^+ \psi^+ \rangle| \leq o(\varepsilon^0) \|\check{H}_f^{1/2} \psi^+\|^2,$$

we arrive at

$$\begin{aligned} & |\langle \psi^+ | (H_{V,m,0}^{\text{np}} - H_{V,m,\varepsilon}^{\text{np}}) \psi^+ \rangle| \\ & \leq (\mathcal{O}(\Delta_*(a, m, \varepsilon)) + o(\varepsilon^0)) \{ \| |D_{\mathbf{A}_m^>}|^{1/2} \psi^+ \|^2 + \| e^F \check{H}_f \psi^+ \|^2 \}, \end{aligned}$$

for $\psi^+ \in \text{Ran}(\mathbb{1}_{(-\infty, \lambda]}(H_{V,m}^\sharp))$. Here we further have

$$2\|e^F \check{H}_f \psi^+\|^2 \leq \|e^{2F} \psi^+\|^2 + \|\check{H}_f^2 \psi^+\|^2,$$

where the norms on the right side can be controlled by our exponential localization and higher order estimates, respectively. Part (ii) is derived analogously. The dependence on λ , the ionization thresholds, and the ground state energies of the constants on the right hand sides of (6.26) and (6.27) stems from the constants in the exponential localization and higher order estimates. \square

6.5 Continuity of the ionization thresholds and ground state energies

To make use of the bounds of Lemmata 6.5 and 6.7 we still have to verify that the functions $m \mapsto \Sigma_m^\sharp$, $m \mapsto E_{V,m}^\sharp$, and $\varepsilon \mapsto \Sigma_{m,\varepsilon}^\sharp$, $\varepsilon \mapsto E_{V,m,\varepsilon}^\sharp$ are (semi-)continuous at 0. The continuity of $\varepsilon \mapsto E_{V,m,\varepsilon}^\sharp$ will also enter more directly into the proof of the existence of ground states in the next subsection.

Corollary 6.8 *Assume that V fulfills Hypothesis 6.1. Then it follows that $\lim_{m \rightarrow 0} E_{V,m}^\sharp = E_V^\sharp$ and $\lim_{m \rightarrow 0} \Sigma_m^\sharp = \Sigma^\sharp$.*

PROOF: Using Lemma 6.5(i) it is not difficult to derive the bounds $\Sigma_m^\sharp \leq \Sigma^\sharp + o(m^0)$. (Given $\varepsilon > 0$ we pick some $\psi^+ \in \text{Ran}(\mathbb{1}_{[\Sigma^\sharp, \Sigma^\sharp + \varepsilon)}(H_0^\sharp))$, $\|\psi^+\| = 1$, and plug it into the quadratic form of $H_{0,m}^\sharp$. In the case of the no-pair operator we also have to observe that $\|P_{\mathbf{A}_m}^+ \psi^+\| \rightarrow 1$, $m \searrow 0$.) Since this gives an upper bound on Σ_m^\sharp which is uniform, for small m , we can then control the constants on the right hand side of the estimate in Lemma 6.5(ii) to get $\Sigma^\sharp \leq \Sigma_m^\sharp + o(m^0)$ by a similar argument. Since the results of Section 3 provide uniform lower bounds on $E_{V,m}$ we can now employ Lemma 6.5 in a similar fashion to show that $\lim_{m \rightarrow 0} E_{V,m}^\sharp = E_V^\sharp$. \square

In order to compare the ionization thresholds Σ_m^\sharp and $\Sigma_{m,\varepsilon}^\sharp$ we need a different argument since \mathbf{x} -dependent weights are required to control the difference between \mathbf{A}_m^\sharp and $\mathbf{A}_{m,\varepsilon}$ but the spectral subspaces of the free operators $H_{0,m,\varepsilon}^\sharp$ are not localized. Here the essential self-adjointness of $H_{0,m,\varepsilon}^\sharp$ asserted in Theorem 3.7 is helpful. To be able to work in one fixed Hilbert space we set

$$\hat{H}_{0,m,\varepsilon}^{\text{np}} := H_{0,m,\varepsilon}^{\text{np}} + P_{\mathbf{A}_{m,\varepsilon}}^- (|D_{\mathbf{A}_{m,\varepsilon}}| + H_{\text{f},m,\varepsilon}) P_{\mathbf{A}_{m,\varepsilon}}^-. \quad (6.28)$$

Then Lemma 8.3 below implies that $\Sigma_{m,\varepsilon}^{\text{np}} = \inf \sigma[H_{0,m,\varepsilon}^{\text{np}}] = \inf \sigma[\hat{H}_{0,m,\varepsilon}^{\text{np}}]$.

Lemma 6.9 $H_{0,m,\varepsilon}^{\text{PF}} \rightarrow H_{0,m,0}^{\text{PF}}$ and $\hat{H}_{0,m,\varepsilon}^{\text{np}} \rightarrow \hat{H}_{0,m,0}^{\text{np}}$ in the strong resolvent sense, as $\varepsilon \searrow 0$. In particular, $\limsup_{\varepsilon \searrow 0} \Sigma_{m,\varepsilon}^\sharp \leq \Sigma_m^\sharp$.

PROOF: Since all involved operators are essentially self-adjoint on \mathcal{D} it suffices to show that $H_{0,m,\varepsilon}^{\text{PF}} \varphi \rightarrow H_{0,m,0}^{\text{PF}} \varphi$ and $\hat{H}_{0,m,\varepsilon}^{\text{np}} \varphi \rightarrow \hat{H}_{0,m,0}^{\text{np}} \varphi$, for every fixed $\varphi \in \mathcal{D}$. Since φ has only finitely many non-vanishing Fock space components and the latter are compactly supported it is clear that $H_{\text{f},m,\varepsilon} \varphi \rightarrow H_{\text{f},m}^\sharp \varphi$. Furthermore, we write

$$|D_{\mathbf{A}_m^\sharp}| - |D_{\mathbf{A}_{m,\varepsilon}}| = S_{\mathbf{A}_m^\sharp} \boldsymbol{\alpha} \cdot (\mathbf{A}_m^\sharp - \mathbf{A}_{m,\varepsilon}) + (S_{\mathbf{A}_m^\sharp} - S_{\mathbf{A}_{m,\varepsilon}}) D_{\mathbf{A}_{m,\varepsilon}},$$

where $S := D|D|^{-1}$ denotes the sign function, which permits to get

$$\begin{aligned} & \| |D_{\mathbf{A}_m^\sharp}| \varphi - |D_{\mathbf{A}_{m,\varepsilon}}| \varphi \| \leq \| e^{-F} \boldsymbol{\alpha} \cdot (\mathbf{A}_m^\sharp - \mathbf{A}_{m,\varepsilon}) \check{H}_{\text{f}}^{-1/2} \| \| e^F \check{H}_{\text{f}}^{1/2} \varphi \| \\ & + \| (S_{\mathbf{A}_m^\sharp} - S_{\mathbf{A}_{m,\varepsilon}}) \check{H}_{\text{f}}^{-1/2} e^{-F} \| \{ \| e^F \check{H}_{\text{f}}^{1/2} D_0 \varphi \| + \| \check{H}_{\text{f}}^{1/2} \boldsymbol{\alpha} \cdot \mathbf{A}_{m,\varepsilon} e^F \varphi \| \}, \end{aligned}$$

where $\check{H}_{\text{f}} := H_{\text{f},m}^\sharp + E$ and $E \equiv E(e, \Lambda)$ is sufficiently large, independently of ε . Using (6.14), (6.19), and Lemma 9.1 it is now easy to see that $H_{0,m,\varepsilon}^{\text{PF}} \varphi \rightarrow H_{0,m,0}^{\text{PF}} \varphi$.

To show that also $\hat{H}_{0,m,\varepsilon}^{\text{np}} \varphi \rightarrow \hat{H}_{0,m,0}^{\text{np}} \varphi$ it remains to observe that

$$\begin{aligned} & \| P_{\mathbf{A}_m^\sharp}^\pm H_{\text{f}}^\sharp P_{\mathbf{A}_m^\sharp}^\pm \varphi - P_{\mathbf{A}_{m,\varepsilon}}^\pm H_{\text{f}}^\sharp P_{\mathbf{A}_{m,\varepsilon}}^\pm \varphi \| \\ & \leq \| (P_{\mathbf{A}_m^\sharp}^\pm - P_{\mathbf{A}_{m,\varepsilon}}^\pm) \check{H}_{\text{f}}^{-1/2} e^{-F} \| \| e^F \check{H}_{\text{f}}^{3/2} P_{\mathbf{A}_m^\sharp}^\pm \check{H}_{\text{f}}^{-3/2} e^{-F} \| \| e^F \check{H}_{\text{f}}^{3/2} \varphi \| \\ & + \| H_{\text{f}} (P_{\mathbf{A}_m^\sharp}^\pm - P_{\mathbf{A}_{m,\varepsilon}}^\pm) \check{H}_{\text{f}}^{-3/2} e^{-F} \| \| e^F \check{H}_{\text{f}}^{3/2} \varphi \| \xrightarrow{\varepsilon \searrow 0} 0, \end{aligned} \quad (6.29)$$

and

$$\|P_{\mathbf{A}_{m,\varepsilon}}^\pm (H_f^> - H_{f,m,\varepsilon}) P_{\mathbf{A}_{m,\varepsilon}}^\pm \varphi\| \leq \frac{\sqrt{3}\varepsilon}{m} \|H_f^> P_{\mathbf{A}_{m,\varepsilon}}^\pm \check{H}_f^{-1}\| \|\check{H}_f \varphi\| \xrightarrow{\varepsilon \searrow 0} 0. \quad (6.30)$$

In fact, (6.29) is valid because of (6.19), (6.21), and since $e^F \check{H}_f^{3/2} P_{\mathbf{A}_m}^\pm \check{H}_f^{-3/2} e^{-F}$ is well-defined and bounded as a consequence of (2.24) and Lemma 9.3. Moreover, (6.30) holds true since $|\omega - \omega_\varepsilon| \leq \sqrt{3}\varepsilon$ and the photon momenta are $\geq m$ in modulus on $\mathcal{H}_m^>$, and since $\|H_f^> P_{\mathbf{A}_{m,\varepsilon}}^\pm \check{H}_f^{-1}\|$ is bounded uniformly in $\varepsilon > 0$ by Lemma 9.3. \square

Corollary 6.10 *Let $m > 0$ be sufficiently small and assume that V fulfills Hypotheses 6.1 and 6.6. Then $\lim_{\varepsilon \rightarrow 0} E_{V,m,\varepsilon}^\sharp = E_{V,m}^\sharp$.*

PROOF: Since Lemma 6.9 provides upper bounds on $\Sigma_{m,\varepsilon}^\sharp$ and we have lower bounds on the spectra of $H_{V,m,\varepsilon}^\sharp$ which are uniform in m and ε , we can apply Lemma 6.7 and some straightforward variational arguments to prove the assertion. We only point out one subtlety: In order to show that $E_{V,m} \leq E_{V,m,\varepsilon} + o(\varepsilon^0)$ we have to pick a test function in $\text{Ran}(\mathbb{1}_{(-\infty, \lambda]}(H_{V,m,\varepsilon}^\sharp))$, for some $\lambda \in (E_{V,m,\varepsilon}, \Sigma_{m,\varepsilon}^\sharp)$. To ensure that this is possible and in order to have an ε -independent bound on the numbers $(\Sigma_{m,\varepsilon}^\sharp - \lambda)^{-1}$ entering into the constant in (6.27) we need the lower bound (6.25) on the binding energy $\Sigma_{m,\varepsilon}^\sharp - E_{V,m,\varepsilon} \geq c$ which does not depend on ε . Without an estimate on the binding energy we still got $E_{V,m,\varepsilon} \leq E_{V,m} + o(\varepsilon^0)$, but we would not have a useful lower bound on $\Sigma_{m,\varepsilon}^\sharp$. \square

6.6 Proofs of the existence of ground states with mass

The next theorem asserting compactness of spectral projections of $H_{V,m,0}^\sharp$ associated with sufficiently low energies is the final result of this section. As in [4] it is proved by estimating the trace of the spectral projection from above by the trace of some finite rank operator, namely the one in (6.37). This finite rank operator is constructed by means of a suitable restriction of the discretized field energy with discrete spectrum and a harmonic oscillator potential which compactifies the electronic part of the operator. In order to sneak in the harmonic oscillator potential in the proofs below we exploit the exponential localization of low-lying spectral subspaces once more. The latter idea stems from [28, 29] and replaces an argument in [4] that works only for small e and/or Λ .

Theorem 6.11 *Let $e \in \mathbb{R}$, $\Lambda \in (0, \infty)$, and assume that V fulfills Hypotheses 6.1 and 6.6. Define $\chi := \mathbb{1}_{(-\infty, E_{V,m}^\sharp + m/4]}(H_{V,m,0}^\sharp)$ and assume that $m > 0$ is sufficiently small. Then $\text{Tr}\{\chi\} < \infty$. In particular, $E_{V,m}^\sharp$ is an eigenvalue of both $H_{V,m,0}^\sharp$ and $H_{V,m}^\sharp$.*

In the proof of the preceding theorem, which is carried through separately for $\sharp = \text{PF}$ and $\sharp = \text{np}$ below, we shall employ an orthogonal splitting of $\mathcal{K}_m^>$ into subspaces of discrete and fluctuating photon states,

$$\mathcal{K}_m^d := P_\varepsilon \mathcal{K}_m^>, \quad \mathcal{K}_m^f := \mathcal{K}_m^> \ominus \mathcal{K}_m^d.$$

Here P_ε is defined in (6.10). The splitting $\mathcal{K}_m^> = \mathcal{K}_m^d \oplus \mathcal{K}_m^f$ gives rise to an isomorphism

$$L^2(\mathbb{R}^3, \mathbb{C}^4) \otimes \mathcal{F}_b[\mathcal{K}_m^>] \cong (L^2(\mathbb{R}^3, \mathbb{C}^4) \otimes \mathcal{F}_b[\mathcal{K}_m^d]) \otimes \mathcal{F}_b[\mathcal{K}_m^f], \quad (6.31)$$

and we observe that the Dirac operator and the field energy decompose under the above isomorphism as

$$D_{\mathbf{A}_{m,\varepsilon}} \cong D_{\mathbf{A}_{m,\varepsilon}^d} \otimes \mathbb{1}^f, \quad H_{\mathbf{f},m,\varepsilon} = H_{\mathbf{f},m,\varepsilon}^d \otimes \mathbb{1}^f + \mathbb{1}^d \otimes H_{\mathbf{f},m,\varepsilon}^f. \quad (6.32)$$

Here and in the following we designate operators acting in the Fock space factors $\mathcal{F}_b[\mathcal{K}_m^\ell]$, $\ell \in \{d, f\}$, by the corresponding superscript d or f . In fact, the discretized vector potential $\mathbf{A}_{m,\varepsilon}$ acts on the various n -particle sectors in $\mathcal{F}_b[\mathcal{K}_m^>]$ by tensor-multiplying or taking scalar products with elements from \mathcal{K}_m^d (apart from symmetrization and a normalization constant).

For $\ell \in \{d, f\}$, we denote the identity on $\mathcal{F}_b[\mathcal{K}_m^\ell]$ by $\mathbb{1}^\ell$ and the projection onto the vacuum sector in $\mathcal{F}_b[\mathcal{K}_m^\ell]$ by P_{Ω^ℓ} and write $P_{\Omega^\ell}^\perp := \mathbb{1}^\ell - P_{\Omega^\ell}$. The identity on $L^2(\mathbb{R}_x^3, \mathbb{C}^4)$ is denoted as $\mathbb{1}^{\text{el}}$.

The semi-relativistic Pauli-Fierz operator.

PROOF: We prove Theorem 6.11 with $\sharp = \text{PF}$. On account of Lemma 6.7(i) we have $\chi H_{V,m,0}^{\text{PF}} \chi \geq \chi H_{V,m,\varepsilon}^{\text{PF}} \chi - o(\varepsilon^0) \chi$. Using (6.32) and $H_{\mathbf{f},m,\varepsilon}^f P_{\Omega^f} = 0$, we then obtain

$$\begin{aligned} & \chi \{ H_{V,m,0}^{\text{PF}} - E_{V,m}^{\text{PF}} - m/2 \} \chi + o(\varepsilon^0) \chi \\ & \geq \chi \{ [|D_{\mathbf{A}_{m,\varepsilon}^d}| + V + H_{\mathbf{f},m,\varepsilon}^d - E_{V,m}^{\text{PF}} - m/2] \otimes P_{\Omega^f} \} \chi \end{aligned} \quad (6.33)$$

$$+ \chi \{ [|D_{\mathbf{A}_{m,\varepsilon}^d}| + V + H_{\mathbf{f},m,\varepsilon}^d - E_{V,m,\varepsilon}^{\text{PF}}] \otimes P_{\Omega^f}^\perp \} \chi \quad (6.34)$$

$$+ \chi \{ \mathbb{1}^{\text{el}} \otimes \mathbb{1}^d \otimes (H_{\mathbf{f},m,\varepsilon}^f - E_{V,m}^{\text{PF}} + E_{V,m,\varepsilon}^{\text{PF}} - m/2) P_{\Omega^f}^\perp \} \chi. \quad (6.35)$$

In view of (6.32) we have $E_{V,m,\varepsilon}^{\text{PF}} = \inf \sigma[|D_{\mathbf{A}_{m,\varepsilon}^d}| + V + H_{\mathbf{f},m,\varepsilon}^d]$. Therefore, the expression in (6.34) is a non-negative quadratic form. For sufficiently small $\varepsilon > 0$, the expression in (6.35) is a non-negative quadratic form, too, because of $H_{\mathbf{f},m,\varepsilon}^f P_{\Omega^f}^\perp \geq m P_{\Omega^f}^\perp$ and Corollary 6.10. In order to treat the remaining term in (6.33) we write

$$\begin{aligned} & [|D_{\mathbf{A}_{m,\varepsilon}^d}| + V + H_{\mathbf{f},m,\varepsilon}^d] \otimes P_{\Omega^f} \\ & = (\mathbb{1} \otimes P_{\Omega^f}) \{ |D_{\mathbf{A}_{m,\varepsilon}}| + V + H_{\mathbf{f},m,\varepsilon} \} (\mathbb{1} \otimes P_{\Omega^f}) \\ & \geq (\mathbb{1} \otimes P_{\Omega^f}) \{ \varepsilon |D_0| + \varepsilon H_{\mathbf{f},m,\varepsilon} - \text{const}(\varepsilon, m, V, e, \Lambda) \} (\mathbb{1} \otimes P_{\Omega^f}). \end{aligned}$$

In the second step we assumed that $\varepsilon > 0$ is small enough. Altogether, we arrive at

$$\begin{aligned} & \chi \{ H_{V,m,0}^{\text{PF}} - E_{V,m}^{\text{PF}} - m/2 \} \chi + o(\varepsilon^0) \chi + \varepsilon \chi |\mathbf{x}|^2 \chi \\ & \geq \chi \{ (\varepsilon |D_0| + \varepsilon |\mathbf{x}|^2 + \varepsilon H_{f,m,\varepsilon}^d - \text{const}) \otimes P_{\Omega^f} \} \chi \\ & \geq \chi \{ [\varepsilon |D_0| + \varepsilon |\mathbf{x}|^2 + \varepsilon H_{f,m,\varepsilon}^d - \text{const}]_- \otimes P_{\Omega^f} \} \chi, \end{aligned} \quad (6.36)$$

where $[\cdots]_- \leq 0$ denotes the negative part. The crucial point about the previous estimate is that both $|D_0| + |\mathbf{x}|^2$ and $H_{f,m,\varepsilon}^d$ have a purely discrete spectrum as operators in the electron and discrete photon Hilbert spaces. Besides P_{Ω^f} has rank one, of course. (ω_ε has a discrete spectrum *as an operator in* $\mathcal{K}_m^d = P_\varepsilon \mathcal{K}$ because the eigenspace in \mathcal{K}_m^d corresponding to some value attained by ω_ε is finite-dimensional. Using $\omega_\varepsilon \geq m > 0$ it is then easy to see that the spectrum of its second quantization, $H_{f,m,\varepsilon}^d = d\Gamma(\omega_\varepsilon \lfloor \mathcal{K}_m^d)$, is discrete, too.) In particular, we observe that

$$W_{m,\varepsilon}^- := [\varepsilon |D_0| + \varepsilon |\mathbf{x}|^2 + \varepsilon H_{f,m,\varepsilon}^d - \text{const}]_- \otimes P_{\Omega^f} \quad (6.37)$$

is a finite rank operator, for every sufficiently small $\varepsilon > 0$, no matter how large the value of the $(\varepsilon, m, V, e, \Lambda)$ -dependent constant is. As a simple consequence of the exponential localization we further know that $\chi |\mathbf{x}|^2 \chi$ is bounded. Using $\chi \{ H_{V,m,0}^{\text{PF}} - E_{V,m}^{\text{PF}} - m/2 \} \chi \leq -(m/4) \chi$ we obtain the bound $(o(\varepsilon^0) - m/4) \text{Tr}\{\chi\} \geq \text{Tr}\{\chi W_{m,\varepsilon}^- \chi\} > -\infty$ from (6.36). Fixing $\varepsilon > 0$ sufficiently small we conclude $\text{Tr}\{\chi\} < \infty$. \square

The no-pair operator.

PROOF: We prove Theorem 6.11 with $\sharp = \text{np}$. On account of Lemma 6.7(i) we again have $\chi H_{V,m,0}^{\text{np}} \chi \geq \chi H_{V,m,\varepsilon}^{\text{np}} \chi - o(\varepsilon^0) \chi$. In view of (6.32) we have $P_{\mathbf{A}_{m,\varepsilon}}^+ = P_{\mathbf{A}_{m,\varepsilon}^d}^+ \otimes \mathbb{1}^f$ with $P_{\mathbf{A}_{m,\varepsilon}^d}^+ := \mathbb{1}_{[0,\infty)}(D_{\mathbf{A}_{m,\varepsilon}^d})$ and we observe that $H_{V,m,\varepsilon}^{\text{np}}$ decomposes under the isomorphism (6.31) as

$$\begin{aligned} H_{V,m,\varepsilon}^{\text{np}} &= \overline{X_\varepsilon^d \otimes \mathbb{1}^f + P_{\mathbf{A}_{m,\varepsilon}^d}^+ \otimes H_{f,m,\varepsilon}^f}, \\ X_\varepsilon^d &:= P_{\mathbf{A}_{m,\varepsilon}^d}^+ (D_{\mathbf{A}_{m,\varepsilon}^d} + V + H_{f,m,\varepsilon}^d) P_{\mathbf{A}_{m,\varepsilon}^d}^+. \end{aligned} \quad (6.38)$$

Writing $\mathbb{1}^{\text{el}} \otimes \mathbb{1}^d = P_{\mathbf{A}_{m,\varepsilon}^d}^+ + P_{\mathbf{A}_{m,\varepsilon}^d}^-$ and $\mathbb{1}^f = P_{\Omega^f} + P_{\Omega^f}^\perp$ and using (6.38) and $H_{f,m,\varepsilon}^f P_{\Omega^f} = 0$, we obtain

$$\begin{aligned} & \chi \{ H_{V,m,0}^{\text{np}} - E_{V,m}^{\text{np}} - m/2 \} \chi + o(\varepsilon^0) \chi \\ & \geq \chi \{ (X_\varepsilon^d - (E_{V,m}^{\text{np}} + m/2) P_{\mathbf{A}_{m,\varepsilon}^d}^+) \otimes P_{\Omega^f} \} \chi \end{aligned} \quad (6.39)$$

$$+ \chi \{ (X_\varepsilon^d - E_{V,m,\varepsilon}^{\text{np}} P_{\mathbf{A}_{m,\varepsilon}^d}^+) \otimes P_{\Omega^f}^\perp \} \chi \quad (6.40)$$

$$+ \chi \{ P_{\mathbf{A}_{m,\varepsilon}^d}^+ \otimes (E_{V,m,\varepsilon}^{\text{np}} - E_{V,m}^{\text{np}} - m/2 + H_{f,m,\varepsilon}^f) P_{\Omega^f}^\perp \} \chi \quad (6.41)$$

$$- (E_{V,m}^{\text{np}} + m/2) \chi \{ P_{\mathbf{A}_{m,\varepsilon}^d}^- \otimes \mathbb{1}^f \} \chi. \quad (6.42)$$

Next, we observe that $E_{V,m,\varepsilon}^{\text{np}} = \inf \sigma[X_\varepsilon^d]$ and proceed as in the proof of Theorem 6.11 with $\sharp = \text{PF}$: We omit the expression in (6.40) which is a non-negative quadratic form. For sufficiently small $\varepsilon > 0$, the term in (6.41) is non-negative also, since $H_{f,m,\varepsilon}^f P_{\Omega^f}^\perp \geq m P_{\Omega^f}^\perp$ and $\lim_{\varepsilon \rightarrow 0} E_{V,m,\varepsilon}^{\text{np}} = E_{V,m}^{\text{np}}$ by Corollary 6.10. The term in (6.42), is some $o(\varepsilon^0)$ on account of $\chi \{P_{\mathbf{A}_{m,\varepsilon}^d}^- \otimes \mathbb{1}^f\} \chi = \chi (P_{\mathbf{A}_m^+} - P_{\mathbf{A}_{m,\varepsilon}^+}) \chi$, (6.19), and the boundedness of $e^F \tilde{H}_f^{1/2} \chi$. (The latter follows from the exponential localization and higher order estimates.) Putting all these remarks together and applying (3.32) in the last step we obtain

$$\begin{aligned}
& \chi \{H_{V,m,0}^{\text{np}} - E_{V,m}^{\text{np}} - m/2\} \chi + o(\varepsilon^0) \chi \\
& \geq \chi \{ (X_\varepsilon^d - (E_{V,m}^{\text{np}} + m/2) P_{\mathbf{A}_{m,\varepsilon}^d}^+) \otimes P_{\Omega^f} \} \chi \\
& = \chi (\mathbb{1} \otimes P_{\Omega^f}) \{H_{V,m,\varepsilon}^{\text{np}} - (E_{V,m}^{\text{np}} + m/2) P_{\mathbf{A}_{m,\varepsilon}^+}^+\} (\mathbb{1} \otimes P_{\Omega^f}) \chi \\
& \geq \chi P_{\mathbf{A}_{m,\varepsilon}^+}^+ \{ [\varepsilon |D_0| + \varepsilon |\mathbf{x}|^2 + \varepsilon H_{f,m,\varepsilon}^d - \text{const}]_- \otimes P_{\Omega^f} \} P_{\mathbf{A}_{m,\varepsilon}^+}^+ \chi \\
& \quad - \varepsilon \chi P_{\mathbf{A}_{m,\varepsilon}^+}^+ \{ |\mathbf{x}|^2 \otimes P_{\Omega^f} \} P_{\mathbf{A}_{m,\varepsilon}^+}^+ \chi,
\end{aligned}$$

where the constant in the penultimate line again depends on ε , m , V , e , and Λ , and the operator in the last line is bounded due to the localization of χ . We may thus conclude as in the end of the proof of Theorem 6.11 with $\sharp = \text{PF}$. \square

7 Infra-red bounds

The final step in the proof of the existence of ground states is a compactness argument showing that a sequence of normalized ground state eigenfunctions, ϕ_{m_j} , $m_j \searrow 0$, of operators with photon masses m_j contains a strongly convergent subsequence whose limit turns out to be a ground state eigenfunction for the original operator. This compactness argument is explained in the subsequent Section 8. As a preparation we now discuss the infra-red bounds stated in the following proposition. They are proved in [28] for the semi-relativistic Pauli-Fierz operator and in [29] for the no-pair operator starting from a suitable representation of $a(k) \phi_m$. In order not to lengthen the present exposition too much we only outline the proof of the soft photon bound for the semi-relativistic Pauli-Fierz operator in Subsection 7.2. We recall the notation

$$(a(k) \psi)^{(n)}(k_1, \dots, k_n) = (n+1)^{1/2} \psi^{(n+1)}(k, k_1, \dots, k_n), \quad n \in \mathbb{N}_0,$$

almost everywhere, where $\psi = (\psi^{(n)})_{n=0}^\infty \in \mathcal{F}_b[\mathcal{K}]$, and $a(k) \Omega = 0$.

Proposition 7.1 *Let $e \in \mathbb{R}$, $\Lambda \in (0, \infty)$, and assume that V fulfills Hypotheses 6.1 and 6.6.. Then there is a constant, $C > 0$, such that, for all sufficiently small $m > 0$ and every normalized ground state eigenvector, ϕ_m^\sharp , of $H_{V,m}^\sharp$, we have the following soft photon bound,*

$$\|a(k) \phi_m^\sharp\|^2 \leq \mathbb{1}_{\{m \leq |\mathbf{k}| \leq \Lambda\}} \frac{C}{|\mathbf{k}|}, \quad (7.1)$$

for almost every $k = (\mathbf{k}, \lambda) \in \mathbb{R}^3 \times \mathbb{Z}_2$, as well as the following photon derivative bound,

$$\|a(k) \phi_m^\# - a(p) \phi_m^\#\| \leq C |\mathbf{k} - \mathbf{p}| \left(\frac{1}{|\mathbf{k}|^{1/2} |\mathbf{k}_\perp|} + \frac{1}{|\mathbf{p}|^{1/2} |\mathbf{p}_\perp|} \right), \quad (7.2)$$

for almost every $k = (\mathbf{k}, \lambda), p = (\mathbf{p}, \mu) \in \mathbb{R}^3 \times \mathbb{Z}_2$ with $m < |\mathbf{k}| < \Lambda$ and $m < |\mathbf{p}| < \Lambda$.

We remark that the photon derivative bound (7.2) is actually the only place in the whole article where the special choice of the polarization vectors (2.4) enters into the analysis.

7.1 The gauge transformed operator

In order to derive the infra-red bounds (7.1) and (7.2) by the method outlined in Subsection 7.2 it is necessary to pass to a suitable gauge [6, 17]. For otherwise we would end up with a more singular infra-red behavior of their right hand sides. To define an appropriate operator-valued gauge transformation [17] we recall that, for $i, j \in \{1, 2, 3\}$, the components $A_m^{(i)}(\mathbf{x})$ and $A_m^{(j)}(\mathbf{y})$ of the magnetic vector potential at $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ commute in the sense that all their spectral projections commute; see, e.g., Theorem X.43 of [43]. Therefore, it makes sense to define

$$U := \int_{\mathbb{R}^3}^\oplus \mathbb{1}_{\mathbb{C}^4} \otimes U_{\mathbf{x}} d^3 \mathbf{x}, \quad U_{\mathbf{x}} := \prod_{j=1}^3 e^{ix_j A_m^{(j)}(\mathbf{0})}, \quad \mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3.$$

Then the gauge transformed vector potential is given by

$$\tilde{\mathbf{A}}_m(\mathbf{x}) := \mathbf{A}_m(\mathbf{x}) - \mathbf{A}_m(\mathbf{0}) = a^\dagger(\tilde{\mathbf{G}}_m) + a(\tilde{\mathbf{G}}_m),$$

where $a^\dagger(\tilde{\mathbf{G}}_m) = \int_{\mathbb{R}^3}^\oplus \mathbb{1}_{\mathbb{C}^4} \otimes a^\dagger(\tilde{\mathbf{G}}_{\mathbf{x},m}) d^3 \mathbf{x}$, and

$$\tilde{\mathbf{G}}_{\mathbf{x},m}(k) := -e \frac{\mathbb{1}_{\{m \leq |\mathbf{k}| \leq \Lambda\}}}{2\pi \sqrt{|\mathbf{k}|}} (e^{i\mathbf{k} \cdot \mathbf{x}} - 1) \boldsymbol{\varepsilon}(k) = (e^{i\mathbf{k} \cdot \mathbf{x}} - 1) \mathbf{G}_{\mathbf{0},m}(k),$$

for all $\mathbf{x} \in \mathbb{R}^3$ and almost every $k = (\mathbf{k}, \lambda) \in \mathbb{R}^3 \times \mathbb{Z}_2$. Here $\mathbf{G}_{\mathbf{x},m}$ is defined in (6.1). In fact, using $[U, \boldsymbol{\alpha} \cdot \mathbf{A}_m] = 0$ we deduce that

$$U D_{\mathbf{A}_m} U^* = D_{\tilde{\mathbf{A}}_m}, \quad U P_{\mathbf{A}_m}^+ U^* = P_{\tilde{\mathbf{A}}_m}^+, \quad U |D_{\mathbf{A}_m}| U^* = |D_{\tilde{\mathbf{A}}_m}|.$$

The crucial point observed in [6] is that the transformed vector potential $\tilde{\mathbf{A}}_m$ has a better infra-red behavior than \mathbf{A}_m in view of the estimate

$$|\tilde{\mathbf{G}}_{\mathbf{x},m}(k)| \leq |\mathbf{k}| |\mathbf{x}| |\mathbf{G}_{\mathbf{0},m}(k)|. \quad (7.3)$$

In particular, infra-red divergent (for $m \searrow 0$) integrals appearing in the derivation of the soft photon bound are avoided when we work with $\tilde{\mathbf{A}}_m$ instead of \mathbf{A}_m . We further set

$$\begin{aligned}\tilde{H}_f &:= U H_f U^* = H_f + i\mathbf{x} \cdot (a(\omega \mathbf{G}_{0,m}) - a^\dagger(\omega \mathbf{G}_{0,m})) \\ &\quad + 2 \langle \omega \mathbf{x} \cdot \mathbf{G}_{0,m} | \mathbf{x} \cdot \mathbf{G}_{0,m} \rangle, \\ \tilde{H}_{V,m}^{\text{PF}} &:= U H_{V,m}^{\text{PF}} U^* = |D_{\tilde{\mathbf{A}}_m}| + V + \tilde{H}_f, \\ \tilde{H}_{V,m}^{\text{np}} &:= U H_{V,m}^{\text{np}} U^* = P_{\tilde{\mathbf{A}}_m}^+ (D_{\tilde{\mathbf{A}}_m} + V + \tilde{H}_f) P_{\tilde{\mathbf{A}}_m}^+, \\ \tilde{\phi}_m^\sharp &:= U \phi_m^\sharp,\end{aligned}\tag{7.4}$$

so that $\tilde{\phi}_m^\sharp$ is a ground state eigenfunction of $\tilde{H}_{V,m}^\sharp$. One can easily show that, if the infra-red bounds (7.1) and (7.2) hold true with ϕ_m^\sharp replaced by $\tilde{\phi}_m^\sharp$, then they are valid for ϕ_m^\sharp as well (with a different constant, of course).

7.2 Soft photon bound for the semi-relativistic Pauli-Fierz operator

To simplify the notation we write $\tilde{\phi}_m$ instead of $\tilde{\phi}_m^{\text{PF}}$ in this subsection. A brief calculation yields

$$\begin{aligned}0 &\leq \langle (\tilde{H}_{V,m}^{\text{PF}} - E_{V,m}^{\text{PF}}) a(k) \tilde{\phi}_m | a(k) \tilde{\phi}_m \rangle = \langle [\tilde{H}_{V,m}^{\text{PF}}, a(k)] \tilde{\phi}_m | a(k) \tilde{\phi}_m \rangle \\ &= \langle [S_{\tilde{\mathbf{A}}_m}, a(k)] D_{\tilde{\mathbf{A}}_m} \tilde{\phi}_m | a(k) \tilde{\phi}_m \rangle + \langle S_{\tilde{\mathbf{A}}_m} [D_{\tilde{\mathbf{A}}_m}, a(k)] \tilde{\phi}_m | a(k) \tilde{\phi}_m \rangle \\ &\quad + \langle [\tilde{H}_f, a(k)] \tilde{\phi}_m | a(k) \tilde{\phi}_m \rangle.\end{aligned}\tag{7.6}$$

Combining the commutation relations

$$[a(k), a^\dagger(f)] = f(k), \quad [a(k), a(f)] = 0, \quad [a(k), H_f] = \omega(k) a(k).$$

with the formula (7.4) for \tilde{H}_f we obtain

$$[\tilde{H}_f, a(k)] = -\omega(k) a(k) + i\omega(k) \mathbf{x} \cdot \mathbf{G}_{0,m}(k).$$

Furthermore, we get $[D_{\tilde{\mathbf{A}}_m}, a(k)] = -\boldsymbol{\alpha} \cdot \tilde{\mathbf{G}}_{\mathbf{x},m}(k)$ and

$$[S_{\tilde{\mathbf{A}}_m}, a(k)] = \int_{-\infty}^{\infty} R_{\tilde{\mathbf{A}}_m}(iy) [a(k), D_{\tilde{\mathbf{A}}_m}] R_{\tilde{\mathbf{A}}_m}(iy) \frac{dy}{\pi},\tag{7.7}$$

where $R_{\tilde{\mathbf{A}}_m}(iy) = (D_{\tilde{\mathbf{A}}_m} - iy)^{-1}$. Next, we insert (7.7) into (7.6), move the term containing $\omega(k) a(k)$ to the left hand side, and divide by $\omega(k)$. Furthermore, we introduce a weight function, $F(\mathbf{x}) = a|\mathbf{x}|^2 / \sqrt{1 + |\mathbf{x}|^2}$, for some small $a > 0$. Abbreviating

$$D_{\tilde{\mathbf{A}}_m}^F := e^F D_{\tilde{\mathbf{A}}_m} e^{-F} = D_{\tilde{\mathbf{A}}_m} + i\boldsymbol{\alpha} \cdot \nabla F, \quad R_{\tilde{\mathbf{A}}_m}^F(iy) := (D_{\tilde{\mathbf{A}}_m}^F - iy)^{-1},$$

we obtain the following result,

$$\begin{aligned}
\|a(k) \tilde{\phi}_m\|^2 &\leq \int_{-\infty}^{\infty} \langle R_{\tilde{\mathbf{A}}_m}(iy) \{ \boldsymbol{\alpha} \cdot \tilde{\mathbf{G}}_{\mathbf{x},m} e^{-F} |\mathbf{k}|^{-1} \} \times \\
&\quad \times R_{\tilde{\mathbf{A}}_m}^F(iy) D_{\tilde{\mathbf{A}}_m}^F e^F \tilde{\phi}_m | a(k) \tilde{\phi}_m \rangle \frac{dy}{\pi} \\
&\quad - \langle S_{\tilde{\mathbf{A}}_m} \{ \boldsymbol{\alpha} \cdot \tilde{\mathbf{G}}_{\mathbf{x},m} e^{-F} |\mathbf{k}|^{-1} \} e^F \tilde{\phi}_m | a(k) \tilde{\phi}_m \rangle \\
&\quad - i \mathbf{G}_{\mathbf{0},m}(k) \cdot \langle (\mathbf{x} e^{-F}) e^F \tilde{\phi}_m | a(k) \tilde{\phi}_m \rangle. \tag{7.8}
\end{aligned}$$

The purpose of the exponentials e^{-F} introduced above is to control the factor $|\mathbf{x}|$ coming from (7.3). In fact,

$$\| \boldsymbol{\alpha} \cdot \tilde{\mathbf{G}}_{\mathbf{x},m} e^{-F} \| \leq \sqrt{2} \sup_{\mathbf{x}} |\tilde{\mathbf{G}}_{\mathbf{x},m} e^{-F(\mathbf{x})}| \leq \mathcal{O}(1) |\mathbf{k}|^{1/2} \mathbb{1}_{\{m \leq |\mathbf{k}| \leq \Lambda\}}.$$

Using this it is easy to see that the sum of the last two expressions in (7.8) is not greater than $\mathcal{O}(1) |\mathbf{k}|^{-1} \mathbb{1}_{\{m \leq |\mathbf{k}| \leq \Lambda\}} \|e^F \tilde{\phi}_m\|^2 + \|a(k) \tilde{\phi}_m\|^2/2$. By virtue of the second resolvent identity we further get

$$R_{\tilde{\mathbf{A}}_m}^F(iy) D_{\tilde{\mathbf{A}}_m}^F = (1 - R_{\tilde{\mathbf{A}}_m}^F(iy) (i\boldsymbol{\alpha} \cdot \nabla F)) R_{\tilde{\mathbf{A}}_m}(iy) (D_{\tilde{\mathbf{A}}_m} + i\boldsymbol{\alpha} \cdot \nabla F).$$

Since we have $\|R_{\tilde{\mathbf{A}}_m}^F(iy)\|, \|R_{\tilde{\mathbf{A}}_m}(iy)\| \leq \mathcal{O}(1) (1 + |y|)^{-1}$ by (2.21) and

$$\|R_{\tilde{\mathbf{A}}_m}(iy) |D_{\tilde{\mathbf{A}}_m}|^{1/2}\| \leq \mathcal{O}(1) (1 + |y|)^{-1/2},$$

we thus obtain

$$\|R_{\tilde{\mathbf{A}}_m}^F(iy) D_{\tilde{\mathbf{A}}_m}^F e^F \tilde{\phi}_m\| \leq \mathcal{O}(1) (1 + |y|)^{-1/2} \| |D_{\tilde{\mathbf{A}}_m}|^{1/2} e^F \tilde{\phi}_m \|.$$

Therefore, the integral in (7.8) converges absolutely and we arrive at

$$\|a(k) \tilde{\phi}_m\|^2 \leq \mathcal{O}(1) |\mathbf{k}|^{-1} \mathbb{1}_{\{m \leq |\mathbf{k}| \leq \Lambda\}} \| |D_{\tilde{\mathbf{A}}_m}|^{1/2} e^F \tilde{\phi}_m \|^2 + 3 \|a(k) \tilde{\phi}_m\|^2/4.$$

If $a > 0$ is small enough, then $\| |D_{\tilde{\mathbf{A}}_m}|^{1/2} e^F \tilde{\phi}_m \|$ is bounded uniformly in $m > 0$, due to a strengthened version of our exponential localization estimates; see Lemma 5.4 of [28] whose proof works for every potential V fulfilling Hypothesis 6.1. Together with the last remark in the preceding subsection this yields the soft photon bound.

The above calculations illustrate the importance of the formal gauge invariance of our models. In fact, without the gauge invariance we could perform these calculations only for $\mathbf{G}_{\mathbf{x},m}$ instead of $\tilde{\mathbf{G}}_{\mathbf{x},m}$. In this case we would, however, end up with an upper bound on $\|a(k) \phi_m\|^2$ of the form $\mathcal{O}(1) |\mathbf{k}|^{-3} \mathbb{1}_{\{m \leq |\mathbf{k}| \leq \Lambda\}}$, which is not integrable near zero and, hence, is not suitable for the arguments presented in the following section.

8 Existence of ground states

We have now collected all prerequisites to show that $\inf \sigma[H_V^\sharp]$ is an eigenvalue of H_V^\sharp . The final compactness argument is presented in the first subsection below. The main idea behind it is borrowed from [17]: Namely, to show that $\{\phi_{m_j}^\sharp\}_{j \in \mathbb{N}}$, $m_j \searrow 0$, contains strongly convergent subsequences we may restrict our attention to finitely many Fock space sectors and to a compact subset of the \mathbf{k} -space. In fact, this is a consequence of the soft photon bounds. On account of the exponential localization estimates it further suffices to consider compact subsets of the \mathbf{x} -space. Moreover, the photon derivative bounds and the localization in energy lead to bounds on the (half-)derivatives of the vectors $\phi_{m_j}^\sharp$ w.r.t. \mathbf{x} and \mathbf{k} on compact sets and in the finitely many Fock space sectors. The idea proposed in [17] is to exploit such information by applying suitable compact embedding theorems. Essentially, we only have to replace the Rellich-Kondrashov theorem applied there by a suitable embedding theorem for spaces of functions with fractional derivatives. (In the non-relativistic case the ground states ϕ_m^\sharp possess weak derivatives with respect to the electron coordinates, whereas in our case we only have Inequality (8.5) below. For a variant of the argument that avoids the Nikol'skiĭ spaces introduced below by switching the roles of the electronic position and momentum spaces see [25].)

In the second subsection we discuss the degeneracy of the ground state energies by applying Kramers' degeneracy theorem similarly as in [40].

In the whole section the coupling function $\mathbf{G}_\mathbf{x}$ is the physical one defined in (2.2).

8.1 Ground states without photon mass

We shall apply the following elementary fact:

Let S be some non-negative operator acting in some Hilbert space, \mathcal{X} . Let $\{\eta_j\}_{j \in \mathbb{N}}$ be some sequence in \mathcal{X} , converging weakly to some $\eta \in \mathcal{X}$ such that $\eta_j \in \mathcal{Q}(S)$ and $\langle \eta_j | S \eta_j \rangle = \|S^{1/2} \eta_j\|^2 \rightarrow 0$. Then η belongs to the domain of S and $S \eta = 0$.

In fact, the linear functional $f(\phi) = \langle \eta | S^{1/2} \phi \rangle$, $\phi \in \mathcal{D}(S^{1/2})$, satisfies

$$f(\phi) = \lim_{j \rightarrow \infty} \langle \eta_j | S^{1/2} \phi \rangle = \lim_{j \rightarrow \infty} \langle S^{1/2} \eta_j | \phi \rangle = 0,$$

and the self-adjointness of $S^{1/2}$ implies $\eta \in \mathcal{D}(S^{1/2})$ and $S^{1/2} \eta = 0$.

We remind the reader of the notation $\sharp \in \{\text{np}, \text{PF}\}$.

Theorem 8.1 *Let $e \in \mathbb{R}$, $\Lambda > 0$, and assume that V fulfills Hypotheses 6.1 and 6.6. Then E_V^\sharp is an eigenvalue of H_V^\sharp .*

PROOF: For every sufficiently small $m > 0$, there is some normalized ground state eigenfunction, ϕ_m^\sharp , of $H_{V,m}^\sharp$. We thus find some sequence $m_j \searrow 0$ such that $\{\phi_{m_j}^\sharp\}_{j \in \mathbb{N}}$ converges weakly to some $\phi^\sharp \in \mathcal{H}$. According to Theorem 3.4 the form domains of H_V^{PF} and $H_{V,m}^{\text{PF}}$, $m > 0$, coincide whence $\phi_{m_j}^{\text{PF}} \in \mathcal{Q}(H_V^{\text{PF}})$. By

the characterization of the form domain of the no-pair operator in Theorem 3.6 we further know that $P_{\mathbf{A}}^+ \phi_{m_j}^{\text{np}} \in \mathcal{Q}(H_V^{\text{np}})$. It is trivial that $P_{\mathbf{A}}^+ \phi_{m_j}^{\text{np}}$ converges weakly to $P_{\mathbf{A}}^+ \phi^{\text{np}}$ in $P_{\mathbf{A}}^+ \mathcal{H}$. Since $N_j := (P_{\mathbf{A}}^+ - P_{\mathbf{A}_{m_j}}^-) \check{H}_{\mathbf{f}}^{-1/2}$ converges to zero in norm, where $\check{H}_{\mathbf{f}} = H_{\mathbf{f}} + E$, for some sufficiently large $E \geq 1$, we further have $\langle \psi | P_{\mathbf{A}}^- \phi^{\text{np}} \rangle = \lim_{j \rightarrow \infty} \langle N_j \check{H}_{\mathbf{f}}^{1/2} \psi | \phi_{m_j}^{\text{np}} \rangle = 0$, for every $\psi \in \mathcal{D}$, whence $P_{\mathbf{A}}^- \phi^{\text{np}} = 0$, or, $P_{\mathbf{A}}^+ \phi^{\text{np}} = \phi^{\text{np}}$. Hence, by the remark preceding the statement it suffices to show that $\phi^\sharp \neq 0$ and $\langle \phi_{m_j}^\sharp | (H_V^\sharp - E_V^\sharp) \phi_{m_j}^\sharp \rangle \rightarrow 0$.

The latter condition immediately follows from $\lim_{m \searrow 0} E_{V,m} = E_V$ and Lemma 6.5(ii) which together imply

$$\langle \phi_{m_j}^\sharp | (H_V^\sharp - E_V^\sharp) \phi_{m_j}^\sharp \rangle = \langle \phi_{m_j}^\sharp | (H_{V,m_j}^\sharp - E_{V,m_j}^\sharp) \phi_{m_j}^\sharp \rangle + o(m_j^0) = o(m_j^0).$$

In order to verify that $\phi^\sharp \neq 0$ we adapt the ideas of [17] sketched in the first paragraph of this section.

We write $\phi_m^\sharp = (\phi_{m,\sharp}^{(n)})_{n=0}^\infty \in \bigoplus_{n=0}^\infty \mathcal{F}_{\mathbf{b}}^{(n)}[\mathcal{K}]$ in what follows. Let $\varepsilon > 0$. By virtue of the soft photon bound we find $n_0 \in \mathbb{N}$ and $C \in (0, \infty)$ such that, for all sufficiently small $m > 0$,

$$\sum_{n=n_0}^\infty \|\phi_{m,\sharp}^{(n)}\|^2 \leq \frac{1}{n_0} \sum_{n=0}^\infty n \|\phi_{m,\sharp}^{(n)}\|^2 = \frac{1}{n_0} \int \|a(k) \phi_m^\sharp\|^2 dk \leq \frac{C}{n_0} < \frac{\varepsilon}{2}. \quad (8.1)$$

By the exponential localization estimates of Theorem 5.1, which hold uniformly for small $m > 0$, we further find some $R > 0$ such that

$$\int_{|\mathbf{x}| \geq R/2} \|\phi_m^\sharp\|_{\mathbb{C}^4 \otimes \mathcal{F}_{\mathbf{b}}}^2(\mathbf{x}) d^3 \mathbf{x} < \frac{\varepsilon}{2}. \quad (8.2)$$

In addition, the soft photon bound ensures that $\phi_{m,\sharp}^{(n)}(\mathbf{x}, \varsigma, k_1, \dots, k_n) = 0$, for almost every $(\mathbf{x}, \varsigma, k_1, \dots, k_n) \in \mathbb{R}^3 \times \{1, 2, 3, 4\} \times (\mathbb{R}^3 \times \mathbb{Z}_2)^n$, $k_j = (\mathbf{k}_j, \lambda_j)$, such that $|\mathbf{k}_j| > \Lambda$, for some $j \in \{1, \dots, n\}$. (Here and henceforth ς labels the four spinor components.) For $0 < n < n_0$ and some fixed $\bar{\theta} = (\varsigma, \lambda_1, \dots, \lambda_n) \in \{1, 2, 3, 4\} \times \mathbb{Z}_2^n$ we set

$$\phi_{m,\bar{\theta},\sharp}^{(n)}(\mathbf{x}, \mathbf{k}_1, \dots, \mathbf{k}_n) := \phi_{m,\sharp}^{(n)}(\mathbf{x}, \varsigma, \mathbf{k}_1, \lambda_1, \dots, \mathbf{k}_n, \lambda_n)$$

and similarly for $\phi^\sharp = (\phi_\sharp^{(n)})_{n=0}^\infty$. Moreover, we set, for every $\delta \geq 0$,

$$Q_{n,\delta} := \{(\mathbf{x}, \mathbf{k}_1, \dots, \mathbf{k}_n) : |\mathbf{x}| < R - \delta, \delta < |\mathbf{k}_j| < \Lambda - \delta, j = 1, \dots, n\}.$$

Fixing some small $\delta > 0$ we pick some cut-off function $\chi \in C_0^\infty(\mathbb{R}^{3(n+1)}, [0, 1])$ such that $\chi \equiv 1$ on $Q_{n,2\delta}$ and $\text{supp}(\chi) \subset Q_{n,\delta}$ and define $\psi_{m,\bar{\theta},\sharp}^{(n)} := \chi \phi_{m,\bar{\theta},\sharp}^{(n)}$. As a next step the photon derivative bound is used to show that $\{\psi_{m,\bar{\theta},\sharp}^{(n)}\}_{m \in (0,\delta]}$

is a bounded family in the anisotropic Nikol'skiĭ space² $H_{\mathbf{q}}^{\mathbf{s}}(\mathbb{R}^{3(n+1)})$, where $\mathbf{s} = (1/2, 1/2, 1/2, 1, \dots, 1)$ and $\mathbf{q} = (2, 2, 2, p, \dots, p)$ with $p \in [1, 2)$. In fact, employing the Hölder inequality (w.r.t. $d^3 \mathbf{x} d^{3(n-1)} \mathbf{K}$) and the photon derivative bound (7.2), we obtain as in [17], for $p \in [1, 2)$, $m \in (0, \delta]$, and $\mathbf{h} \in \mathbb{R}^3$,

$$\begin{aligned} & \int_{\substack{Q_{n,\delta} \cap \\ \{\delta < |\mathbf{k} + \mathbf{h}| < \Lambda\}}} |\phi_{m,\bar{\theta},\#}^{(n)}(\mathbf{x}, \mathbf{k} + \mathbf{h}, \mathbf{K}) - \phi_{m,\bar{\theta},\#}^{(n)}(\mathbf{x}, \mathbf{k}, \mathbf{K})|^p d^3 \mathbf{x} d^3 \mathbf{k} d^{3(n-1)} \mathbf{K} \\ & \leq C \sum_{\lambda \in \mathbb{Z}_2} \int_{\substack{m < |\mathbf{k}| < \Lambda, \\ m < |\mathbf{k} + \mathbf{h}| < \Lambda}} \|a(\mathbf{k} + \mathbf{h}, \lambda) \phi_m^\# - a(\mathbf{k}, \lambda) \phi_m^\#\|_{L^p}^p d^3 \mathbf{k} \leq C' |\mathbf{h}|^p, \end{aligned}$$

where the constants $C, C' \in (0, \infty)$ do not depend on $m \in (0, \delta]$. Since $\phi_{m,\bar{\theta},\#}^{(n)}$ is symmetric in the photon variables the previous estimate implies [42, §4.8] that the weak first order partial derivatives of $\phi_{m,\bar{\theta}}^{(n)}$ with respect to its last $3n$ variables exist on $Q_{n,\delta}$ and that

$$\|\phi_{m,\bar{\theta},\#}^{(n)}\|_{W_p^{\mathbf{r}}(Q_{n,\delta})}^p := \|\phi_{m,\bar{\theta},\#}^{(n)}\|_{L^p(Q_{n,\delta})}^p + \sum_{j=1}^n \sum_{i=1}^3 \|\partial_{k_j^{(i)}} \phi_{m,\bar{\theta},\#}^{(n)}\|_{L^p(Q_{n,\delta})}^p \leq C'',$$

for $m \in (0, \delta]$ and some m -independent $C'' \in (0, \infty)$, with $\mathbf{r} := (0, 0, 0, 1, \dots, 1)$. The previous estimate implies $\|\psi_{m,\bar{\theta},\#}^{(n)}\|_{W_p^{\mathbf{r}}(\mathbb{R}^{3(n+1)})} \leq C'''$, for some $C''' \in (0, \infty)$ which does not depend on $m \in (0, \delta]$. Moreover, the anisotropic Sobolev space $W_p^{\mathbf{r}}(\mathbb{R}^{3(n+1)})$ is continuously embedded into $H_p^{\mathbf{r}}(\mathbb{R}^{3(n+1)})$; see, e.g., [42, §6.2]. By Theorems 3.4 and 3.6 we get, for $n \in \mathbb{N}$,

$$c^{-1} \langle \phi_{m,\#}^{(n)} | |D_0| \phi_{m,\#}^{(n)} \rangle \leq \langle \phi_{m,\#} | H_{V,m}^\# \phi_{m,\#} \rangle + c = E_{V,m}^\# + c \leq E_V^\# + 2c, \quad (8.5)$$

for some m -independent $c \in (0, \infty)$. Therefore, $\{\phi_{m,\bar{\theta},\#}^{(n)}\}_{m \in (0, \delta]}$ and, hence, $\{\psi_{m,\bar{\theta},\#}^{(n)}\}_{m \in (0, \delta]}$ are bounded families in the Bessel potential, or, Liouville space $L_2^{\mathbf{r}'}(\mathbb{R}^{3(n+1)})$, $\mathbf{r}' := (1/2, 1/2, 1/2, 0, \dots, 0)$, where the fractional derivatives are

² For $r_1, \dots, r_d \in [0, 1]$, $q_1, \dots, q_d \geq 1$, we have $H_{q_1, \dots, q_d}^{(r_1, \dots, r_d)}(\mathbb{R}^d) := \bigcap_{i=1}^d H_{q_i x_i}^{r_i}(\mathbb{R}^d)$. For $r_i \in [0, 1)$, a measurable function $f : \mathbb{R}^d \rightarrow \mathbb{C}$ belongs to the class $H_{q_i x_i}^{r_i}(\mathbb{R}^d)$, if $f \in L^{q_i}(\mathbb{R}^d)$ and there is some $M \in (0, \infty)$ such that

$$\|f(\cdot + h \mathbf{e}_i) - f\|_{L^{q_i}(\mathbb{R}^d)} \leq M |h|^{r_i}, \quad h \in \mathbb{R}, \quad (8.3)$$

where \mathbf{e}_i is the i -th canonical unit vector in \mathbb{R}^d . If $r_i = 1$, then (8.3) is replaced by

$$\|f(\cdot + h \mathbf{e}_i) - 2f + f(\cdot - h \mathbf{e}_i)\|_{L^{q_i}(\mathbb{R}^d)} \leq M |h|, \quad h \in \mathbb{R}. \quad (8.4)$$

$H_{q_1, \dots, q_d}^{(r_1, \dots, r_d)}(\mathbb{R}^d)$ is a Banach space with norm

$$\|f\|_{q_1, \dots, q_d}^{(r_1, \dots, r_d)} := \max_{1 \leq i \leq d} \|f\|_{L^{q_i}(\mathbb{R}^d)} + \max_{1 \leq i \leq d} M_i,$$

where M_i is the infimum of all constants $M > 0$ satisfying (8.3) or (8.4), respectively. Finally, we abbreviate $H_q^{(r_1, \dots, r_d)}(\mathbb{R}^d) := H_{q, \dots, q}^{(r_1, \dots, r_d)}(\mathbb{R}^d)$.

defined by means of the Fourier transform. The embedding $L_2'(\mathbb{R}^{3(n+1)}) \rightarrow H_2'(\mathbb{R}^{3(n+1)})$ is continuous, too; see §9.3 in [42]. Altogether it follows that $\{\psi_{m,\bar{\theta},\#}^{(n)}\}_{m \in (0,\delta]}$ is a bounded family in $H_{\mathbf{q}}^s(\mathbb{R}^{3(n+1)})$. Now, we may apply Theorem 3.2 in [41]. The latter ensures that $\{\psi_{m,\bar{\theta},\#}^{(n)}\}_{m \in (0,\delta]}$ contains a sequence which is strongly convergent in $L^2(Q_{n,2\delta})$ provided $1 - 3n(p^{-1} - 2^{-1}) > 0$. Of course, we can choose $p < 2$ large enough such that the latter condition is fulfilled, for all $n = 1, \dots, n_0 - 1$. By finitely many repeated selections of subsequences we may hence assume without loss of generality that $\{\phi_{m_j,\bar{\theta},\#}^{(n)}\}_{j \in \mathbb{N}}$ converges strongly in $L^2(Q_{n,2\delta})$ to its weak limit $\phi_{\bar{\theta},\#}^{(n)}$, for $0 \leq n < n_0$. In particular, by the choice of n_0 and R in (8.1) and (8.2),

$$\|\phi^\sharp\|^2 \geq \lim_{j \rightarrow \infty} \sum_{n=0}^{n_0-1} \sum_{\bar{\theta}} \|\phi_{m_j,\bar{\theta},\#}^{(n)}\|_{L^2(Q_{n,2\delta})}^2 \geq \lim_{j \rightarrow \infty} \|\phi_{m_j}^\sharp\|^2 - \varepsilon - o(\delta^0),$$

where we use the soft photon bound to estimate

$$\begin{aligned} & \sum_{n=1}^{n_0-1} \sum_{\bar{\theta}} \left\| \phi_{m_j,\bar{\theta},\#}^{(n)} \mathbb{1}_{\{\exists i : |\mathbf{k}_i| \leq 2\delta \vee |\mathbf{k}_i| \geq \Lambda - 2\delta\}} \right\|^2 \\ & \leq \sum_{\lambda \in \mathbb{Z}_2} \int_{\substack{\{|\mathbf{k}| \leq 2\delta\} \cup \\ \{|\mathbf{k}| \geq \Lambda - 2\delta\}}} \|a(\mathbf{k}, \lambda) \phi_{m_j}^\sharp\|^2 d^3 \mathbf{k} = o(\delta^0), \quad \delta \searrow 0. \end{aligned}$$

Hence, $\|\phi^\sharp\|^2 \geq 1 - \varepsilon - o(\delta^0)$, where $\delta > 0$ and $\varepsilon > 0$ are arbitrary, that is, $\|\phi^\sharp\| = 1$. (In particular, $\phi_{m_j}^\sharp \rightarrow \phi^\sharp$ strongly in \mathcal{H} .) \square

8.2 Ground state degeneracy

Suppose that $V(\mathbf{x}) = V(-\mathbf{x})$. As already mentioned in the introduction it is remarked in [40] that every (speculative) eigenvalue of H_V^{PF} and, in particular, its ground state energy is evenly degenerate in this case. The authors prove this statement by constructing some anti-linear involution commuting with H_V^{PF} and applying Kramers' degeneracy theorem. We shall do the same for the no-pair operator in the next theorem which originates from [29].

Theorem 8.2 *Let $e \in \mathbb{R}$, $\Lambda > 0$, assume that V fulfills Hypotheses 6.1, and assume in addition that $V(\mathbf{x}) = V(-\mathbf{x})$, for almost every \mathbf{x} . If the ground state energy E_V^{np} is an eigenvalue of H_V^{np} , then it is evenly degenerate.*

PROOF: Similarly as in [40] we introduce the anti-linear operator

$$\vartheta := J \alpha_2 C R = -\alpha_2 J C R, \quad J := \begin{pmatrix} 0 & \mathbb{1}_2 \\ -\mathbb{1}_2 & 0 \end{pmatrix},$$

where $C : \mathcal{H} \rightarrow \mathcal{H}$ denotes complex conjugation, $C\psi := \overline{\psi}$, $\psi \in \mathcal{H}$, and $R : \mathcal{H} \rightarrow \mathcal{H}$ is the parity transformation $(R\psi)(\mathbf{x}) := \psi(-\mathbf{x})$, for almost every

$\mathbf{x} \in \mathbb{R}^3$ and every $\psi \in \mathcal{H} = L^2(\mathbb{R}_{\mathbf{x}}^3, \mathbb{C}^4 \otimes \mathcal{F}_{\mathbf{b}}[\mathcal{H}])$. Obviously, $[\vartheta, -i\partial_{x_j}] = [\vartheta, V] = [\vartheta, H_{\mathbf{f}}] = 0$, on $\mathcal{D}(|D_{\mathbf{0}}|) \cap \mathcal{D}(H_{\mathbf{f}})$. Since α_2 squares to one and $C\alpha_2 = -\alpha_2 C$, for all entries of α_2 are purely imaginary, we further get $\vartheta^2 = -\mathbb{1}$ and $[\vartheta, \alpha_2] = 0$. Moreover, the Dirac matrices α_0 , α_1 , and α_3 have real entries and $[J\alpha_2, \alpha_j] = J\{\alpha_2, \alpha_j\} = 0$ by (2.1), whence $[\vartheta, \alpha_j] = 0$, for $j \in \{0, 1, 3\}$. Finally $[\vartheta, e^{\pm i\mathbf{k} \cdot \mathbf{x}}] = 0$ implies $[\vartheta, A^{(j)}] = 0$ on $\mathcal{D}(H_{\mathbf{f}}^{1/2})$, for $j \in \{1, 2, 3\}$. It follows that $[\vartheta, D_{\mathbf{A}}] = 0$ on $\mathcal{D} = \vartheta \mathcal{D}$ and, since $D_{\mathbf{A}}$ is essentially self-adjoint on \mathcal{D} , we obtain $\vartheta \mathcal{D}(D_{\mathbf{A}}) = \mathcal{D}(D_{\mathbf{A}})$ and $[\vartheta, D_{\mathbf{A}}] = 0$ on $\mathcal{D}(D_{\mathbf{A}})$, which implies $\vartheta R_{\mathbf{A}}(iy) - R_{\mathbf{A}}(-iy)\vartheta = 0$ on \mathcal{H} , for every $y \in \mathbb{R}$. Using the representation (2.18) we conclude that $[\vartheta, P_{\mathbf{A}}^+] = 0$ on \mathcal{H} . In particular, ϑ can be considered as an operator acting on $\mathcal{H}_{\mathbf{A}}^+$. Furthermore, we obtain $H_V^{\text{np}} \vartheta \varphi - \vartheta H_V^{\text{np}} \varphi = 0$, for every $\varphi \in \mathcal{D}$. Since $P_{\mathbf{A}}^+ \mathcal{D}$ is a form core for H_V^{np} we can easily extend this commutation relation and show that ϑ maps $\mathcal{D}(H_V^{\text{np}})$ into itself and $H_V^{\text{np}} \vartheta \psi = \vartheta H_V^{\text{np}} \psi$, for every $\psi \in \mathcal{D}(H_V^{\text{np}})$. Hence, by Kramers' degeneracy theorem every eigenvalue of H_V^{np} is evenly degenerated. (In fact, $H_V^{\text{np}} \phi = E_V \phi$ implies $H_V^{\text{np}} \vartheta \phi = E_V \vartheta \phi$, and $\phi \perp \vartheta \phi$, since $\langle \vartheta \phi | \phi \rangle = -\langle \vartheta \phi | \vartheta(\vartheta \phi) \rangle = -\langle C \phi | C \vartheta \phi \rangle = -\langle \vartheta \phi | \phi \rangle$.) \square

Using similar arguments we derive the next lemma which has been referred to in Sections 5 and 6. We use the notation introduced in Subsection 6.1 and write

$$H_{0,m,\varepsilon}^{\text{np},-} := P_{\mathbf{A}_{m,\varepsilon}}^- (|D_{\mathbf{A}_{m,\varepsilon}}| + H_{\mathbf{f},m,\varepsilon}) P_{\mathbf{A}_{m,\varepsilon}}^-$$

for the analogs of the no-pair operators on the negative spectral subspace which appeared in (5.1) and (6.28). Then $\hat{H}_{0,m,\varepsilon}^{\text{np}} = H_{0,m,\varepsilon}^{\text{np}} + H_{0,m,\varepsilon}^{\text{np},-}$; see (5.1) and (6.28). We unify the notation by setting $H_{0,0,0}^{\text{np}} := H_0^{\text{np}}$, etc.

Lemma 8.3 *Let $e \in \mathbb{R}$, $\Lambda \in (0, \infty)$, and $m \geq \varepsilon \geq 0$. Then*

$$\Sigma_{m,\varepsilon}^{\text{np}} \stackrel{\text{def.}}{=} \inf \sigma[H_{0,m,\varepsilon}^{\text{np}}] = \inf \sigma[H_{0,m,\varepsilon}^{\text{np},-}] = \inf \sigma[\hat{H}_{0,m,\varepsilon}^{\text{np}}].$$

PROOF: Let C and R be defined as in the preceding proof. We introduce the anti-linear operator $\tau : \mathcal{H}_m^> \rightarrow \mathcal{H}_m^>$, $\tau := \alpha_2 C R$. Similar as in the previous proof we verify that $\tau D_{\mathbf{A}_{m,\varepsilon}} = -D_{\mathbf{A}_{m,\varepsilon}} \tau$ on $\mathcal{D}(D_{\mathbf{A}_{m,\varepsilon}})$, thus $R_{\mathbf{A}_{m,\varepsilon}}(iy) \tau = -\tau R_{\mathbf{A}_{m,\varepsilon}}(iy)$, thus $P_{\mathbf{A}_{m,\varepsilon}}^{\pm} \tau = \tau P_{\mathbf{A}_{m,\varepsilon}}^{\mp}$ again by (2.18). Notice that we use the property (6.9) of the discretized phase in $\mathbf{A}_{m,\varepsilon}$ to obtain these relations in the case $\varepsilon > 0$. Consequently, $H_{0,m,\varepsilon}^{\text{np}} \tau = \tau H_{0,m,\varepsilon}^{\text{np},-}$ on a natural dense domain (again called \mathcal{D}) in $\mathcal{H}_m^>$ with $\tau \mathcal{D} = \mathcal{D}$. By Theorem 3.7 we may assume that the no-pair operators in the plus or minus spaces are essentially self-adjoint on $P_{\mathbf{A}_{m,\varepsilon}}^{\pm} \mathcal{D}$, respectively, and we readily conclude. \square

9 Commutator estimates

In this final section we collect some bounds proved in [37] on the operator norm of various commutators which have been used repeatedly in the preceding sections. In the whole section we assume that $\mathbf{G}_{\mathbf{x}}$ fulfills Hypothesis 2.1.

9.1 Basic estimates

Our estimates on commutators involving the field energy H_f are based on the next lemma. The following quantity appears in its statement and in various estimates below,

$$\delta_\nu^2 \equiv \delta_\nu(E)^2 := 8 \int \frac{w_\nu(k, E)^2}{\omega(k)} \|\mathbf{G}(k)\|_\infty^2 dk, \quad E, \nu > 0, \quad (9.1)$$

where $w_\nu(k, E) := E^{1/2-\nu} ((E+\omega(k))^{\nu+1/2} - E^\nu (E+\omega(k))^{1/2})$. We observe that $w_{1/2}(k, E) \leq \omega(k)$ and, hence, $\delta_{1/2}(E) \leq 2d_1$, $E > 0$. Moreover, $\delta_\nu(E) \leq \delta_\nu(1)$, for $E \geq 1$. We recall the identity

$$\langle H_f^{1/2} \phi | H_f^{1/2} \psi \rangle = \int \omega(k) \langle a(k) \phi | a(k) \psi \rangle dk, \quad \phi, \psi \in \mathcal{D}(H_f^{1/2}), \quad (9.2)$$

which is a consequence of the permutation symmetry and Fubini's theorem.

Lemma 9.1 *Let ν , $E > 0$, and set $\check{H}_f := H_f + E$. Then*

$$\| [\boldsymbol{\alpha} \cdot \mathbf{A}, \check{H}_f^{-\nu}] \check{H}_f^\nu \| \leq \delta_\nu(E)/E^{1/2}. \quad (9.3)$$

PROOF: ([37]) We pick $\phi, \psi \in \mathcal{D}$ and write

$$\begin{aligned} & \langle \phi | [\boldsymbol{\alpha} \cdot \mathbf{A}, \check{H}_f^{-\nu}] \check{H}_f^\nu \psi \rangle \\ &= \langle \phi | [\boldsymbol{\alpha} \cdot a(\mathbf{G}), \check{H}_f^{-\nu}] \check{H}_f^\nu \psi \rangle - \langle [\boldsymbol{\alpha} \cdot a(\mathbf{G}), \check{H}_f^{-\nu}] \phi | \check{H}_f^\nu \psi \rangle. \end{aligned} \quad (9.4)$$

By definition of $a(k)$ and H_f we have the pull-through formula $a(k) \theta(H_f) \psi = \theta(H_f + \omega(k)) a(k) \psi$, for almost every k and every Borel function θ on \mathbb{R} , which leads to

$$\begin{aligned} & [a(k), \check{H}_f^{-\nu}] \check{H}_f^\nu \psi \\ &= \{ ((\check{H}_f + \omega(k))^{-\nu} - \check{H}_f^{-\nu}) (\check{H}_f + \omega(k))^{\nu+1/2} \} a(k) \check{H}_f^{-1/2} \psi. \end{aligned}$$

We denote the operator $\{\dots\}$ by $F(k)$. Then $F(k)$ is bounded and

$$\begin{aligned} \|F(k)\| &\leq \int_0^1 \sup_{t \geq 0} \left| \frac{d}{ds} \frac{(t + E + \omega(k))^{\nu+1/2}}{(t + E + s\omega(k))^\nu} \right| ds \\ &= - \int_0^1 \frac{d}{ds} \frac{(E + \omega(k))^{\nu+1/2}}{(E + s\omega(k))^\nu} ds = w_\nu(k, E)/E^{1/2}. \end{aligned}$$

Using these remarks, the Cauchy-Schwarz inequality, and (9.2), we obtain

$$\begin{aligned} & |\langle \phi | [\boldsymbol{\alpha} \cdot a(\mathbf{G}), \check{H}_f^{-\nu}] \check{H}_f^\nu \psi \rangle| \\ &\leq \int \|\phi\| \|\boldsymbol{\alpha} \cdot \mathbf{G}(k)\| \|F(k)\| \|a(k) \check{H}_f^{-1/2} \psi\| dk \\ &\leq \|\phi\| \left(2 \int \frac{\|F(k)\|^2}{\omega(k)} \|\mathbf{G}(k)\|_\infty^2 dk \right)^{1/2} \left(\int \omega(k) \|a(k) \check{H}_f^{-1/2} \psi\|^2 dk \right)^{1/2} \\ &\leq \frac{\delta_\nu(E)}{2E^{1/2}} \|\phi\| \|H_f^{1/2} \check{H}_f^{-1/2} \psi\|. \end{aligned}$$

A similar argument applied to the second term in (9.4) yields

$$|\langle [\boldsymbol{\alpha} \cdot \mathbf{a}(\mathbf{G}), \check{H}_f^{-\nu}] \phi | \check{H}_f^\nu \psi \rangle| \leq \frac{\tilde{\delta}_\nu(E)}{2E^{1/2}} \|H_f^{1/2} \check{H}_f^{-1/2} \phi\| \|\psi\|,$$

where $\tilde{\delta}_\nu(E)$ is defined by (9.1) with $w_\nu(k, E)$ replaced by

$$\tilde{w}_\nu(k, E) := E^{1/2-\nu} (E^\nu (E + \omega(k))^{1/2} - E^{2\nu} (E + \omega)^{1/2-\nu}).$$

Evidently, $\tilde{w}_\nu \leq w_\nu$, thus $\tilde{\delta}_\nu \leq \delta_\nu$, which concludes the proof. \square

Choosing E large enough, we can certainly make to norm in (9.3) as small as we please. This observation can be exploited to ensure that certain Neumann series converge in the next proof which yields convenient formulas allowing to interchange the field energy with resolvents of the Dirac operator.

We define $J : [0, 1) \rightarrow \mathbb{R}$ by $J(0) := 1$ and $J(a) := \sqrt{6}/(1-a^2)$, for $a \in (0, 1)$, so that $\|R_{\mathbf{A}}^F(iy)\| \leq J(a)(1+y^2)^{-1/2}$, where $R_{\mathbf{A}}^0(iy) := R_{\mathbf{A}}(iy)$; recall (2.20) and (2.21).

Corollary 9.2 *Let $y \in \mathbb{R}$ and $F \in C^\infty(\mathbb{R}_{\mathbf{x}}^3, \mathbb{R})$ such that $|\nabla F| \leq a < 1$. Assume that $\nu, E > 0$ satisfy $\delta_\nu J(a)/E^{1/2} < 1$, and introduce the following operators, $T_\nu := [\check{H}_f^{-\nu}, \boldsymbol{\alpha} \cdot \mathbf{A}] \check{H}_f^\nu$,*

$$\Xi_\nu^F(iy) := \sum_{j=0}^{\infty} \{-R_{\mathbf{A}}^F(iy) T_\nu\}^j, \quad \Upsilon_\nu^F(iy) := \sum_{j=0}^{\infty} \{-T_\nu^* R_{\mathbf{A}}^F(iy)\}^j. \quad (9.5)$$

Then $\|T_\nu\| \leq \delta_\nu/E^{1/2}$, $\|\Theta_\nu^F(iy)\| \leq (1 - \delta_\nu J(a)/E^{1/2})^{-1}$, $\Theta \in \{\Xi, \Upsilon\}$, and

$$\check{H}_f^{-\nu} R_{\mathbf{A}}^F(iy) = \Xi_\nu^F(iy) R_{\mathbf{A}}^F(iy) \check{H}_f^{-\nu}, \quad (9.6)$$

$$R_{\mathbf{A}}^F(iy) \check{H}_f^{-\nu} = \check{H}_f^{-\nu} R_{\mathbf{A}}^F(iy) \Upsilon_\nu^F(iy), \quad (9.7)$$

In particular, $R_{\mathbf{A}}^F(iy)$ maps $\mathcal{D}(H_f^\nu)$ into itself.

PROOF: The following somewhat formal computations show the simple idea behind the proof (see Section 3 of [37] for more details),

$$\begin{aligned} \check{H}_f^{-\nu} R_{\mathbf{A}}^F(iy) &= R_{\mathbf{A}}^F(iy) \check{H}_f^{-\nu} + R_{\mathbf{A}}^F(iy) [D_{\mathbf{A}} + i\boldsymbol{\alpha} \cdot \nabla F - iy, \check{H}_f^{-\nu}] R_{\mathbf{A}}^F(iy) \\ &= R_{\mathbf{A}}^F(iy) \check{H}_f^{-\nu} + R_{\mathbf{A}}^F(iy) [\boldsymbol{\alpha} \cdot \mathbf{A}, \check{H}_f^{-\nu}] \check{H}_f^\nu (\check{H}_f^{-\nu} R_{\mathbf{A}}^F(iy)), \end{aligned}$$

which implies that $R_{\mathbf{A}}^F(iy) \check{H}_f^{-\nu} = (1 + R_{\mathbf{A}}^F(iy) T_\nu)^{-1} \check{H}_f^{-\nu} R_{\mathbf{A}}^F(iy)$. The operator inverse appearing here is given by the Neumann series $\Xi_\nu^F(iy)$ when we choose E in $\check{H}_f = H_f + E$ so large that $\delta_\nu J(a)/E^{1/2} < 1$. \square

9.2 Commuting projections with the field energy

Lemma 9.3 *Let $a, \kappa \in [0, 1)$, let ν, E , and F be as in Corollary 9.2, and assume that F is bounded. Define $\mathcal{C}_\nu^F := e^F [S_{\mathbf{A}}, \check{H}_f^{-\nu}] \check{H}_f^\nu e^{-F}$ on $\mathcal{D}(H_f^\nu)$. Then*

$$\| |D_{\mathbf{A}}|^\kappa \mathcal{C}_\nu^F \| \leq (1 + a J(a)) \frac{\text{const}(\kappa) \delta_\nu J(a)/E^{1/2}}{1 - \delta_\nu J(a)/E^{1/2}}. \quad (9.8)$$

In particular, $S_{\mathbf{A}}$ maps $\mathcal{D}(H_{\mathbf{f}}^\nu)$ into itself and, if $E^{1/2} > \delta_\nu$, then the following identities hold true on $\mathcal{D}(H_{\mathbf{f}}^\nu)$, where $\mathcal{C}_\nu := (\mathcal{C}_\nu^0)^* \in \mathcal{L}(\mathcal{H})$,

$$\check{H}_{\mathbf{f}}^\nu S_{\mathbf{A}} = S_{\mathbf{A}} \check{H}_{\mathbf{f}}^\nu + \mathcal{C}_\nu \check{H}_{\mathbf{f}}^\nu, \quad S_{\mathbf{A}} \check{H}_{\mathbf{f}}^\nu = \check{H}_{\mathbf{f}}^\nu S_{\mathbf{A}} + \check{H}_{\mathbf{f}}^\nu \mathcal{C}_\nu^*. \quad (9.9)$$

PROOF: ([37]) Combining (9.6) with (2.18) we obtain, for all $\phi \in \mathcal{D}(|D_{\mathbf{A}}|^\kappa)$ and $\psi \in \mathcal{D}(H_{\mathbf{f}}^\nu)$,

$$\begin{aligned} |\langle |D_{\mathbf{A}}|^\kappa \phi | \mathcal{C}_\nu^F \psi \rangle| &\leq \int_{\mathbb{R}} |\langle |D_{\mathbf{A}}|^\kappa \phi | R_{\mathbf{A}}^F(iy) T_\nu \Xi_\nu^F(iy) R_{\mathbf{A}}^F(iy) \psi \rangle| \frac{dy}{\pi} \\ &\leq \|T_\nu\| \int_{\mathbb{R}} \| |D_{\mathbf{A}}|^\kappa R_{\mathbf{A}}^F(iy) \phi \| \|\Xi_\nu^F(iy)\| \|R_{\mathbf{A}}^F(iy)\| \frac{dy}{\pi} \|\phi\| \|\psi\|. \end{aligned}$$

Here we estimate $\|T_\nu\|$ by means of (9.3) and we write

$$|D_{\mathbf{A}}|^\kappa R_{\mathbf{A}}^F(iy) = |D_{\mathbf{A}}|^\kappa R_{\mathbf{A}}(iy) (\mathbb{1} - i\alpha \cdot \nabla F R_{\mathbf{A}}^F(iy)),$$

where $\| |D_{\mathbf{A}}|^\kappa R_{\mathbf{A}}(iy) \| \leq \text{const}(\kappa)(1 + y^2)^{-1/2+\kappa/2}$. Moreover, $\|\Xi_\nu^F(iy)\| \leq (1 - \delta_\nu J(a)/E^{1/2})^{-1}$, $y \in \mathbb{R}$, by Corollary 9.2. Altogether these remarks yield the asserted estimate. Now, the identity $S_{\mathbf{A}} \check{H}_{\mathbf{f}}^{-\nu} = \check{H}_{\mathbf{f}}^{-\nu} S_{\mathbf{A}} - \check{H}_{\mathbf{f}}^{-\nu} (\mathcal{C}_\nu^0)^*$ in $\mathcal{L}(\mathcal{H})$ shows that $S_{\mathbf{A}}$ maps the domain of $H_{\mathbf{f}}^\nu$ into itself and that the first identity in (9.9) is valid. Taking the adjoint of (9.9) and using $[\check{H}_{\mathbf{f}}^\nu S_{\mathbf{A}}]^* = S_{\mathbf{A}} \check{H}_{\mathbf{f}}^\nu$ (which is true since $\check{H}_{\mathbf{f}}^\nu S_{\mathbf{A}}$ is densely defined and $S_{\mathbf{A}} = S_{\mathbf{A}}^{-1} \in \mathcal{L}(\mathcal{H})$) we also obtain the second identity in (9.9). \square

We can now prove an estimate asserted in the proof of Theorem 3.4, namely $\mathcal{T} = \text{Re } \Theta \leq \varepsilon |D_{\mathbf{A}}| + \varepsilon^{-1} \text{const } d_1/E^{1/2}$, for $\varepsilon \in (0, 1]$ and $E \geq (4d_1)^2$, where $\Theta := [|D_{\mathbf{A}}|, \check{H}_{\mathbf{f}}^{-1/2}] \check{H}_{\mathbf{f}}^{1/2}$. In fact, this follows easily from Corollary 9.2 and Lemma 9.3 since $\Theta = |D_{\mathbf{A}}|^{1/2} S_{\mathbf{A}} \{|D_{\mathbf{A}}|^{1/2} \mathcal{C}_{1/2}^0\} + T_{1/2} \mathcal{C}_{1/2}^0 - T_{1/2} S_{\mathbf{A}}$.

9.3 Double commutators

Lemma 9.4 *Let $a \in [0, 1)$ and let F satisfy (5.8). Moreover, let $\nu, E > 0$ such that $\delta_\nu J(a)/E^{1/2} \leq 1/2$. Then*

$$\| |D_{\mathbf{A}}| [\chi_1 e^F, [P_{\mathbf{A}}^+, \chi_2 e^{-F}]] \| \leq J(a) \prod_{i=1,2} (a + \|\nabla \chi_i\|_\infty), \quad (9.10)$$

$$\| \check{H}_{\mathbf{f}}^\nu [\chi_1 e^F, [P_{\mathbf{A}}^+, \chi_2 e^{-F}]] \check{H}_{\mathbf{f}}^{-\nu} \| \leq 8 J(a) \prod_{i=1,2} (a + \|\nabla \chi_i\|_\infty). \quad (9.11)$$

Moreover, for every self-adjoint multiplication operator V in $L^2(\mathbb{R}^3)$ satisfying $H^1(\mathbb{R}^3) \subset \mathcal{D}(V)$, we find some constant $C_V \in (0, \infty)$, depending only on V , such that

$$\| |V| [\chi_1 e^F, [P_{\mathbf{A}}^+, \chi_2 e^{-F}]] \check{H}_{\mathbf{f}}^{-1/2} \| \leq C_V J(a) \prod_{i=1,2} (a + \|\nabla \chi_i\|_\infty). \quad (9.12)$$

In (9.12) we also assume that $E \geq (4d_1 J(a))^2$ and $E \geq 1$.

PROOF: ([37]) Let $\phi, \psi \in \mathcal{D}$, $\|\phi\| = \|\psi\| = 1$. First, we derive a bound on

$$I_{\phi, \psi} := \int_{\mathbb{R}} \left| \left\langle |D_{\mathbf{A}}| \phi \left| \check{H}_{\mathbf{f}}^{\nu} [\chi_1 e^F, [R_{\mathbf{A}}(iy), \chi_2 e^{-F}]] \check{H}_{\mathbf{f}}^{-\nu} \psi \right\rangle \right| \frac{dy}{2\pi}.$$

Expanding the double commutator we get

$$[\chi_1 e^F, [R_{\mathbf{A}}(iy), \chi_2 e^{-F}]] = \eta(\chi_1, \chi_2, F; y) + \eta(\chi_2, \chi_1, -F; y),$$

where

$$\begin{aligned} & \eta(\chi_1, \chi_2, F; y) \\ &:= R_{\mathbf{A}}(iy) \boldsymbol{\alpha} \cdot (\nabla \chi_1 + \chi_1 \nabla F) e^F R_{\mathbf{A}}(iy) e^{-F} \boldsymbol{\alpha} \cdot (\nabla \chi_2 - \chi_2 \nabla F) R_{\mathbf{A}}(iy). \end{aligned}$$

We obtain

$$\begin{aligned} & \int_{\mathbb{R}} \left| \left\langle |D_{\mathbf{A}}| \phi \left| \check{H}_{\mathbf{f}}^{\nu} \eta(\chi_1, \chi_2, F; y) \check{H}_{\mathbf{f}}^{-\nu} \psi \right\rangle \right| \frac{dy}{2\pi} \\ & \leq \int_{\mathbb{R}} \left| \left\langle \phi \left| |D_{\mathbf{A}}| R_{\mathbf{A}}(iy) \Upsilon_{\nu}^0(iy) \boldsymbol{\alpha} \cdot (\nabla \chi_1 + \chi_1 \nabla F) \times \right. \right. \right. \\ & \quad \left. \left. \left. \times R_{\mathbf{A}}^F(iy) \Upsilon_{\nu}^F(iy) \boldsymbol{\alpha} \cdot (\nabla \chi_2 - \chi_2 \nabla F) R_{\mathbf{A}}(iy) \Upsilon_{\nu}^0(iy) \psi \right\rangle \right| \frac{dy}{2\pi} \\ & \leq \frac{(a + \|\nabla \chi_1\|)(a + \|\nabla \chi_2\|)}{(1 - \delta_{\nu}/E^{1/2})^2} \cdot \frac{J(a)}{1 - \delta_{\nu} J(a)/E^{1/2}} \int_{\mathbb{R}} \frac{dy}{2\pi(1 + y^2)}. \end{aligned} \quad (9.13)$$

A bound analogous to (9.13) holds true when the roles of χ_1 and χ_2 are interchanged and F is replaced by $-F$. Consequently, $I_{\phi, \psi}$ is bounded by two times the right hand side of (9.13). Altogether this shows that (9.10) and (9.11) hold true. (Just ignore $|D_{\mathbf{A}}|$ or $\check{H}_{\mathbf{f}}$, respectively, in the above argument.) By the closed graph theorem we further have $\| |V| \psi \|^2 \leq \text{const}(V) (\|\nabla \psi\|^2 + \|\psi\|^2)$, $\psi \in H^1(\mathbb{R}^3)$. Hence, (9.12) follows from (9.10), (9.11), and the inequality

$$\| |V| \varphi \|^2 \leq \text{const}(V) \left(\| |D_{\mathbf{A}}| \varphi \|^2 + \| \check{H}_{\mathbf{f}}^{1/2} \varphi \|^2 \right), \quad \varphi \in \mathcal{D},$$

which holds true, for $E \geq d_1^2$, by (3.2), (3.23), and

$$\| |V| \varphi \|^2 \leq \text{const}(V) (\|\nabla \varphi\|^2 + \|\varphi\|^2), \quad \varphi \in \mathcal{D}.$$

□

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References

- [1] Y. Avron, I. Herbst, and B. Simon, “Schrödinger operators with magnetic fields. I. General interactions”, *Duke Math. J.*, **45** (1978), 847–883.
- [2] V. Bach, T. Chen, J. Fröhlich, and I. M. Sigal, “Smooth Feshbach map and operator-theoretic renormalization group methods”, *J. Funct. Anal.*, **203** (2003), 44–92.
- [3] V. Bach, J. Fröhlich, and A. Pizzo, “Infrared-finite algorithms in QED: the groundstate of an atom interacting with the quantized radiation field”, *Comm. Math. Phys.*, **264** (2006), 145–165.
- [4] V. Bach, J. Fröhlich, and I. M. Sigal, “Quantum electrodynamics of confined nonrelativistic particles”, *Adv. Math.*, **137** (1998), 299–395.
- [5] V. Bach, J. Fröhlich, and I. M. Sigal, “Renormalization group analysis of spectral problems in quantum field theory”, *Adv. Math.*, **137** (1998), 205–298.
- [6] V. Bach, J. Fröhlich, and I. M. Sigal, “Spectral analysis for systems of atoms and molecules coupled to the quantized radiation field”, *Comm. Math. Phys.*, **207** (1999), 249–290.
- [7] V. Bach and M. Könenberg, “Construction of the ground state in nonrelativistic QED by continuous flows”, *J. Differential Equations*, **231** (2006), 693–713.
- [8] J.-M. Barbaroux, M. Dimassi, and J.-C. Guillot, “Quantum electrodynamics of relativistic bound states with cutoffs”, *J. Hyperbolic Differ. Equ.*, **1** (2004), 271–314.
- [9] A. Berthier and V. Georgescu, “On the point spectrum of Dirac operators”, *J. Funct. Anal.*, **71** (1987), 309–338.
- [10] K. T. Cheng, M. H. Chen, and W. R. Johnson, “Accurate relativistic calculations including QED contributions for few-electron systems”, In: P. Schwerdtfeger (Editor). *Relativistic electronic structure theory. Part 2: Applications*. Theoretical and Computational Chemistry, **14**, Pages 120–187, Elsevier, 2002.
- [11] E. B. Davies, *Spectral theory and differential operators*, Cambridge Studies in Advanced Mathematics, **42**, Cambridge University Press, Cambridge, 1995.
- [12] M. Dimassi and J. Sjöstrand, *Spectral asymptotics in the semi-classical limit*, London Math. Soc. Lecture Note Series, **268**, Cambridge University Press, Cambridge, 1999.

- [13] W. D. Evans, P. Perry, and H. Siedentop, “The spectrum of relativistic one-electron atoms according to Bethe and Salpeter”, *Comm. Math. Phys.*, **178** (1996), 733–746.
- [14] J. Fröhlich, M. Griesemer, and B. Schlein, “Asymptotic electromagnetic fields in models of quantum-mechanical matter interacting with the quantized radiation field”, *Adv. Math.*, **164** (2001), 349–398.
- [15] J. Fröhlich, M. Griesemer, and I. M. Sigal, “On spectral renormalization group”, *Rev. Math. Phys.*, **21** (2009), 511–548.
- [16] M. Griesemer, “Exponential decay and ionization thresholds in non-relativistic quantum electrodynamics”, *J. Funct. Anal.*, **210** (2004), 321–340.
- [17] M. Griesemer, E. H. Lieb, and M. Loss, “Ground states in non-relativistic quantum electrodynamics”, *Invent. Math.*, **145** (2001), 557–595.
- [18] M. Griesemer and C. Tix, “Instability of a pseudo-relativistic model of matter with self-generated magnetic field”, *J. Math. Phys.*, **40** (1999), 1780–1791.
- [19] F. Hiroshima, “Diamagnetic inequalities for systems of nonrelativistic particles with a quantized field”, *Rev. Math. Phys.*, **8** (1996), 185–203.
- [20] F. Hiroshima, “Functional integral representation of a model in quantum electrodynamics”, *Rev. Math. Phys.*, **9** (1997), 489–530.
- [21] F. Hiroshima and I. Sasaki, “On the ionization energy of the semi-relativistic Pauli-Fierz model for a single particle”, *RIMS Kokyuroku Bessatsu*, **21** (2010), 25–34.
- [22] W. R. Johnson, “Relativistic many-body perturbation theory for highly charged ions”, In: J. J. Boyle and M. S. Pindzola (Editors). *Many-body atomic physics*. Pages 39–64, University Press, 1998.
- [23] T. Kato, *Perturbation theory for linear operators*, Classics in Mathematics, Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.
- [24] M. Könenberg, *Nichtexistenz von Grundzuständen für minimal an das quantisierte Strahlungsfeld gekoppelte, pseudorelativistische Modelle*, Diploma Thesis, Universität Mainz, 2004.
- [25] M. Könenberg and O. Matte, “Ground states of semi-relativistic Pauli-Fierz and no-pair Hamiltonians in QED at critical Coulomb coupling”, *J. Operator Theory*, to appear.
- [26] M. Könenberg and O. Matte, “The mass shell in the semi-relativistic Pauli-Fierz model”, *Preprint*, arXiv:1204.5123v1, 2012, 44 pages.

- [27] M. Könenberg and O. Matte, “On enhanced binding and related effects in the non- and semi-relativistic Pauli-Fierz models”,
- [28] M. Könenberg, O. Matte, and E. Stockmeyer, “Existence of ground states of hydrogen-like atoms in relativistic quantum electrodynamics I: The semi-relativistic Pauli-Fierz operator”, *Rev. Math. Phys.*, **23** (2011), 375–407.
- [29] M. Könenberg, O. Matte, and E. Stockmeyer, “Existence of ground states of hydrogen-like atoms in relativistic quantum electrodynamics II: The no-pair operator”, *J. Math. Phys.*, **52** (2011), 123501, 34 pages.
- [30] E. Lieb and M. Loss, *Analysis*, Graduate Studies in Mathematics, American Mathematical Society, Providence, 2001. Second edition.
- [31] E. H. Lieb and M. Loss, “A bound on binding energies and mass renormalization in models of quantum electrodynamics”, *J. Statist. Phys.*, **108** (2002), 1057–1069.
- [32] E. H. Lieb and M. Loss, “Stability of a model of relativistic quantum electrodynamics”, *Comm. Math. Phys.*, **228** (2002), 561–588.
- [33] E. H. Lieb and M. Loss, “Existence of atoms and molecules in non-relativistic quantum electrodynamics”, *Adv. Theor. Math. Phys.*, **7** (2003), 667–710.
- [34] E. H. Lieb, H. Siedentop, and J. P. Solovej, “Stability and instability of relativistic electrons in classical electromagnetic fields”, *J. Statist. Phys.*, **89** (1997), 37–59.
- [35] O. Matte, *Existence of ground states for a relativistic hydrogen atom coupled to the quantized electromagnetic field*, Diploma Thesis, Universität Mainz, 2000.
- [36] O. Matte, “On higher order estimates in quantum electrodynamics”, *Documenta Math.*, **15** (2010), 207–234.
- [37] O. Matte and E. Stockmeyer, “Exponential localization for a hydrogen-like atom in relativistic quantum electrodynamics”, *Comm. Math. Phys.*, **295** (2010), 551–583.
- [38] O. Matte and E. Stockmeyer, “On the eigenfunctions of no-pair operators in classical magnetic fields”, *Integr. equ. oper. theory*, **65** (2009), 255–283.
- [39] O. Matte and E. Stockmeyer, “Spectral theory of no-pair Hamiltonians”, *Rev. Math. Phys.*, **22** (2010), 1–53.
- [40] T. Miyao and H. Spohn, “Spectral analysis of the semi-relativistic Pauli-Fierz Hamiltonian”, *J. Funct. Anal.*, **256** (2009), 2123–2156.

- [41] S. M. Nikol'skiĭ, "An imbedding theorem for functions with partial derivatives considered in different metrics", *Izv. Akad. Nauk SSSR Ser. Mat.*, **22** (1958), 321–336. (Russian.) English translation in: *Amer. Math. Soc. Transl. (2)*, 90:27–43, 1970.
- [42] S. M. Nikol'skiĭ, *Approximation of functions of several variables and imbedding theorems*, Die Grundlehren der Mathematischen Wissenschaften, Band 205, Springer-Verlag, New York, 1975.
- [43] M. Reed and B. Simon, *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*, Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1975.
- [44] M. Reed and B. Simon, *Methods of modern mathematical physics. IV. Analysis of operators*, Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1978.
- [45] M. Reed and B. Simon, *Methods of modern mathematical physics. I. Functional Analysis*, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, second edition, 1980.
- [46] M. Reiher and A. Wolf, *Relativistic quantum chemistry*, Wiley-VCH, Weinheim, 2009.
- [47] B. Simon, "Kato's inequality and the comparison of semigroups", *J. Funct. Anal.*, **32** (1979), 97–101.
- [48] E. Stockmeyer, "On the non-relativistic limit of a model in quantum electrodynamics", *Preprint*, arXiv:0905.1006v1, 2009, 13 pages.
- [49] C. Tix, "Strict positivity of a relativistic Hamiltonian due to Brown and Ravenhall", *Bull. London Math. Soc.*, **30** (1998), 283–290.